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THE AERODYNAMIC FORCES ON AN AEROFOIL IN NON-UNIFORM UNSTEADY MOTION IN A CLOSED TUNNEL

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CONTENTS

The two-dimensional unsteady motion of an aerofoil, situated midway between parallel walls, and moving through an inviscid, incompressible fluid, is investigated. A completely general upwash distribution is taken, and expressions are obtained for the pressure on the aerofoil surface and the lift and moment about the mid-chord point. By a conformal transformation involving Jacobian elliptic functions the physical plane is mapped into a rectangle, and the theory is based on a solution of Laplace's equation satisfying certain given boundary conditions on this rectangle.

Special cases are considered in which the upwash is (*a*) a sudden upgust, and (*b*) a harmonic oscillation. Detailed examination is made of a rigid-body aerofoil performing translational and rotational harmonic oscillations. The aerodynamic forces are expressed in terms of dimensionless 'air-load coefficients', which are then compared with corresponding coefficients for an aerofoil in an infinitely deep stream. The air-load coefficients are obtained in a form which readily enables first-order corrections for wall interference to be evaluated. It is shown that the formulae derived are at variance with corresponding results obtained by other authors using different methods.

1. Introduction

The unsteady motion of an aerofoil through an inviscid, incompressible fluid is affected by the presence of wind-tunnel walls. The constraint due to the walls causes the aerodynamic forces to assume values different from their free-stream values. The purpose of this paper is to obtain expressions for these differences, that is, to find the corrections which must be made to wind-tunnel measurements of the lift and moment on the aerofoil to reduce them to their corresponding free-stream values.

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Solutions for a special case of this problem, namely, harmonic unsteady perturbations of the aerofoil, have been given by Timman (1951) and Lilley (1952). Timman's method was based on the Green function for the field of a source or vortex in the presence of the aerofoil and walls, while Lilley used the well-known vortex-sheet method, which replaces the aerofoil and its wake by a suitable distribution of doublets and vortices. The two solutions are, apart from a number of minor printing errors in the former paper, identical. However, both authors have carried out operations on certain formal Fourier expansions, and the validity of some of these operations is doubtful. As is shown in § 14, the solutions of the present paper for the special case in question do not agree with those of Timman and Lilley.

The general theory below is based on a method developed by Woods (1955*a*) involving an analytic solution of Laplace's equation satisfying certain given boundary conditions. The flow is two-dimensional, and the aerofoil is situated midway between parallel walls. It is assumed that the aerofoil is sufficiently thin for the thickness effects to be of second order, and hence negligible. The problem of the unsteady motion of a *thick* aerofoil in a free stream has been solved by Woods (1954), but the mathematical difficulties encountered are formidable, and would be considerably magnified in the present problem. It is assumed that the aerofoil in its unsteady motion generates an infinitely thin vortex sheet or 'wake' from the trailing edge. It is further assumed that the unsteady motion is of small amplitude, so that to sufficient accuracy the boundary conditions may be applied at the 'mean' position of the unsteady motion, taken here to be the position where the aerofoil and vortex sheet lie parallel to the tunnel walls.

The boundary conditions to be applied are: the flow direction is constant on the walls, is a known function of time on the aerofoil surface, and is the same on either side of the wake, since the latter has zero thickness.

The analysis below follows the pattern of Woods (1955*b*) in solving the problem of an unsteady aerofoil in a free jet. An arbitrary upwash is imposed on the aerofoil and is specified as a function of time and position along the aerofoil surface. General expressions are obtained for the consequent lift and moment about the mid-chord point. The flow direction on the aerofoil surface, which occurs in the formulae for the lift and moment, is shown, to sufficient accuracy, to be directly proportional to the upwash, except in the neighbourhood of the front stagnation point, whose movement has to be taken into account.

The problem is examined in detail for two types of upwash distribution. The first is that of a sudden upgust; that is, the aerofoil in its steady motion suddenly encounters an upgust of fluid perpendicular to the direction of motion, or, equivalently, the aerofoil suddenly acquires a finite increment to its incidence. The Laplace transform technique enables a formal solution to be obtained in this case.

The second type of upwash considered is one which gives rise to oscillations varying harmonically with time. In particular, an examination is made of the case of a rigid body aerofoil performing translational and rotational oscillations. Timman (1951) and Lilley (1952) limited their considerations to this type of motion.

The lift and moment are expressed in terms of dimensionless numbers, commonly known as the air-load coefficients (see § 10). The difference between these coefficients and their corresponding free-stream values will clearly depend on the width of the tunnel. The appropriate parameter is found to be the ratio of aerofoil length to tunnel width. From the exact, general theory, approximate formulae are derived for these air-load coefficients, which are correct to first order in the parameter chosen. The relevant equations and discussion appear in § 13.

2. Basic mathematical theory

Let (q, θ) be the velocity vector in polar co-ordinates of a flow past a thin aerofoil mid-way between tunnel walls, as shown in figure 1. Let U be a standard reference velocity, which may be taken without loss of generality to be the stream velocity at infinity upstream.

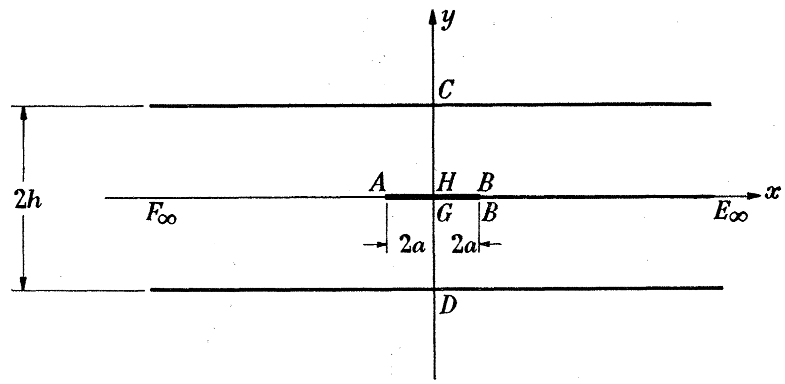


Figure 1. The z plane.

The aerofoil is assumed to be thin enough, and its unsteady perturbations about its mean position to be small enough, to allow us to impose the boundary conditions prevailing on the aerofoil surface over the strip $-2a \leqslant x \leqslant 2a$, $y = 0$ without significant error. Here we have taken the origin of the $z(=x+iy)$ plane to be at the mid-chord point of the aerofoil, and the chord length to be $4a$. The total width of the tunnel is $2h$, so that the tunnel walls are the lines $y = \pm h$. The unsteady motion of the aerofoil will result in a vortex sheet or wake, which we can assume lies on $y = 0$, $2a \leqslant x < \infty$. The validity of this assumption is discussed by Greidanus & van Heemert (1948) for the case of an unsteady aerofoil in an infinite stream, and the argument applies similarly in this problem.

The z plane is mapped into a rectangle in the $t(=\gamma+i\eta)$ plane, shown in figure 2, by a transformation involving Jacobian elliptic functions. The transformation is (see Timman 1951; Woods 1955 *a*, *b*)

$$\operatorname{cn}(t, k) = -\frac{k'}{k}\sinh\frac{\pi z}{2h}, \tag{1}$$

where the moduli k and k' of the elliptic functions are given by

$$\left.\begin{aligned} k &= \tanh(\pi a/h), \\ \text{or, since } k^2+k'^2 = 1, \qquad \frac{k}{k'} &= \sinh(\pi a/h). \end{aligned}\right\} \tag{2}$$

The real and imaginary quarter periods of the elliptic functions will be denoted, as usual, by K and K' respectively. The theta functions associated with these elliptic functions have parameter τ, which is related to the elliptic periods by

$$\tau = iK'/K \tag{3}$$

(see Whittaker & Watson 1946, p. 479).

In the t plane, the aerofoil 'surface' $y = 0$, $-2a \leqslant x \leqslant 2a$ maps into the line $\eta = 0$, $-2K \leqslant \gamma \leqslant 2K$, the upper and lower surfaces of the vortex sheet map into $\gamma = 2K$, $0 \leqslant \eta \leqslant K'$ and $\gamma = -2K$, $0 \leqslant \eta \leqslant K'$ respectively, while the walls become $\eta = 0$, $-2K \leqslant \gamma \leqslant 2K$. The point upstream at infinity, namely F, becomes $t = \mathrm{i}K'$, while the point downstream at infinity, E, becomes $t = \pm 2K + \mathrm{i}K'$.

Now let $w = \phi + \mathrm{i}\psi$ be the complex stream function of the flow; then

$$\mathrm{d}w/\mathrm{d}z = q\,\mathrm{e}^{-\mathrm{i}\theta}.$$

If we define a function f by

$$f \equiv \ln\left(U\frac{\mathrm{d}z}{\mathrm{d}w}\right) = \ln\left(\frac{U}{q}\right) + \mathrm{i}\theta = \Omega + \mathrm{i}\theta, \tag{4}$$

where

$$\Omega \equiv \ln(U/q), \tag{5}$$

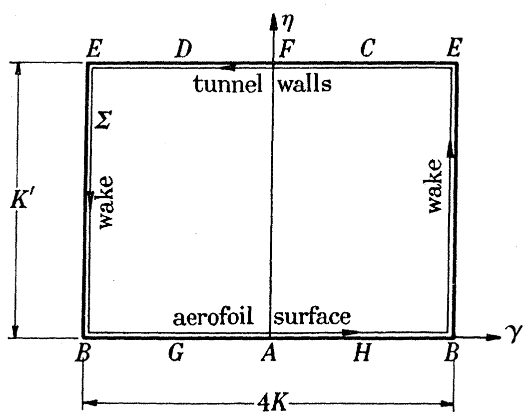


FIGURE 2. The t plane.

then f is an analytic function of z throughout the region $-\infty < x < \infty$, $-h \leqslant y \leqslant h$, except at any stagnation points or sharp corners on the aerofoil surface. Also, because of the conformal nature of the transformation (1), f is an analytic function of t within and on the rectangle $-2K \leqslant \gamma \leqslant 2K$, $0 \leqslant \eta \leqslant K'$, except at corresponding singular points on the line $-2K \leqslant \gamma \leqslant 2K$, $\eta = 0$.

The boundary conditions imposed on the function f are now considered. As mentioned earlier it is assumed here that

(i) the aerofoil is sufficiently thin to be regarded as a flat plate;

(ii) the deviations of the aerofoil from its mean position are of sufficiently small amplitude to allow the application of the boundary conditions at these mean positions without significant error.

The boundary conditions are:

(*a*) the flow direction θ is constant on the walls, and by a suitable selection of axes this constant is taken to be zero, that is, $\theta = 0$ on $\eta = K'$, $-2K \leqslant \gamma \leqslant 2K$;

(*b*) the flow direction is a known function of time on the aerofoil surface $\eta = 0$, $-2K \leqslant \gamma \leqslant 2K$;

(*c*) the jump in the value of the velocity logarithm Ω across the wake, that is,

$$\mathrm{X} \equiv \Omega_{2K} - \Omega_{-2K}, \tag{6}$$

can be calculated directly (see § 5). Since the wake has zero thickness, $\theta_{2K} = \theta_{-2K}$, so that

$$\mathrm{X} = f_{2K} - f_{-2K}.$$

With these boundary conditions, the problem reduces to that of finding a function f satisfying Laplace's equation within and on a rectangle, given the imaginary part of the function on one pair of opposite sides, and the jump in the value of the function across the other pair of opposite sides. A solution of this boundary value problem has been obtained by Woods (1955 *a*). It is

$$f(t) = \Omega_{hm} + \frac{1}{4K}\int_{-2K}^{2K} \theta^* \frac{\vartheta_1'}{\vartheta_1}\left\{\frac{\pi}{4K}(\gamma^*-t) \mid \tau_1\right\} d\gamma^* + \frac{1}{8K}\int_0^{K'} X\left[\frac{\vartheta_2'}{\vartheta_2}\left\{\frac{\pi}{4K}(i\eta^*-t) \mid \tau_1\right\} - \frac{\vartheta_2'}{\vartheta_2}\left\{\frac{\pi}{4K}(i\eta^*+t) \mid \tau_1\right\}\right] d\eta^*, \tag{7}$$

where Ω_{hm} is the 'mean' value of Ω on the walls,

$$\Omega_{hm} \equiv \frac{1}{4K}\int_{-2K}^{2K} \Omega(\gamma + iK')\, d\gamma,$$

θ^* is the value of θ on the aerofoil surface, and X is the jump in Ω across the wake, as defined in equation (6).

The theta functions of equation (7) are designated as in Whittaker & Watson (1946, p. 463) and ϑ' denotes the derivative of a theta function. The parameter τ_1 of these functions is related to the periods of the elliptic functions previously defined by

$$\tau_1 = iK'/2K.$$

The form of the solution (7) is inconvenient for the purposes of the present problem, since the parameter τ_1 of the theta functions is different from the associated parameter τ of the elliptic functions defined in equations (1) to (3). In addition, the theta functions do not here lend themselves as readily to algebraic manipulation as do Jacobian elliptic functions. Hence we transform equation (7) to a form involving elliptic functions and Jacobian zeta functions, having associated parameter $\tau = iK'/K$, as in (3). This alternative form of solution is (see appendix 1)

$$f(t) = \Omega_{hm} + \frac{1}{2\pi}\int_{-2K}^{2K} \theta^*\left[\frac{1+\operatorname{cn}(\gamma^*-t)\operatorname{dn}(\gamma^*-t)}{\operatorname{sn}(\gamma^*-t)} + Z(\gamma^*-t)\right] d\gamma^* + \frac{1}{4\pi}\int_0^{K'} X\left[\frac{\operatorname{cn}(i\eta^*-t)\operatorname{dn}(i\eta^*-t)-1}{\operatorname{sn}(i\eta^*-t)} - \frac{\operatorname{cn}(i\eta^*+t)\operatorname{dn}(i\eta^*+t)-1}{\operatorname{sn}(i\eta^*+t)} + Z(i\eta^*-t) - Z(i\eta^*+t)\right] d\eta^*, \tag{8}$$

where $Z(u)$ is the Jacobian zeta function, as defined in Whittaker & Watson (1946, p. 517).

3. Conditions on the function $f(t)$

There are certain equations involving θ^*, the flow direction on the aerofoil surface, and X, the jump in Ω across the wake, which may be obtained from consideration of (*a*) the analytic character of $f(t)$, and (*b*) the conditions upstream and downstream at infinity.

Consider the integral of $f(t)$ around the closed contour Σ, shown in figure 2. This contour is the perimeter of the rectangle $-2K \leqslant \gamma \leqslant 2K$, $0 \leqslant \eta \leqslant K'$ in the t plane, with appropriate semicircular indentations to exclude singular points on the perimeter. These singularities correspond to any stagnation points or sharp corners on the aerofoil surface, and are logarithmic in character. Hence it may be readily shown that as the radii of these indentations tend to zero they make no contribution to the integral. Thus, since $f(t)$ is analytic throughout the interior of the rectangle, it follows that

$$\int_{\Sigma} f(t)\,\mathrm{d}t = 0.$$

With f replaced by $\Omega+\mathrm{i}\theta$, this may be written as

$$\int_{-2K}^{2K}[\Omega(\gamma)+\mathrm{i}\theta(\gamma)]\,\mathrm{d}\gamma+\mathrm{i}\int_{0}^{K'}[\Omega(2K+\mathrm{i}\eta)+\mathrm{i}\theta(2K+\mathrm{i}\eta)]\,\mathrm{d}\eta$$
$$+\int_{2K}^{-2K}[\Omega(\gamma+\mathrm{i}K')+\mathrm{i}\theta(\gamma+\mathrm{i}K')]\,\mathrm{d}\gamma+\mathrm{i}\int_{K'}^{0}[\Omega(-2K+\mathrm{i}\eta)+\mathrm{i}\theta(-2K+\mathrm{i}\eta)]\,\mathrm{d}\eta = 0.$$

Equating the imaginary parts of this equation gives

$$\int_{-2K}^{2K}\theta^*\,\mathrm{d}\gamma+\int_{0}^{K'}[\Omega(2K+\mathrm{i}\eta)-\Omega(-2K+\mathrm{i}\eta)]\,\mathrm{d}\eta-\int_{-2K}^{2K}\theta(\gamma+\mathrm{i}K')\,\mathrm{d}\gamma = 0,$$

that is,
$$\int_{-2K}^{2K}\theta^*\,\mathrm{d}\gamma^*+\int_{0}^{K'}\mathrm{X}\,\mathrm{d}\eta^* = 0, \tag{9}$$

since $\theta(\gamma+\mathrm{i}K') = 0$.

We now consider the boundary conditions upstream and downstream at infinity, that is, at $z = -\infty, \infty$. Since $\theta = 0$ on the walls, it is clear that $\lim\limits_{z\to\pm\infty}\theta = 0$. Also, by definition of the reference velocity U, it follows that $\lim\limits_{z\to-\infty}\Omega = 0$. Finally, application of the conservation-of-mass principle gives

$$\lim_{z\to-\infty}\Omega = \lim_{z\to\infty}\Omega.$$

Thus we have
$$\lim_{z\to-\infty}f = 0, \quad \lim_{z\to\infty}f = 0.$$

As $z\to-\infty$ corresponds to $t\to\mathrm{i}K'$, and $z\to\infty$ to $t\to\pm 2K+\mathrm{i}K'$, these limits may be written

$$\lim_{t\to\mathrm{i}K'}f = 0, \quad \lim_{t\to\pm2K+\mathrm{i}K'}f = 0.$$

Applying these limits in turn to equation (8), we obtain

$$0 = \Omega_{hm}+\frac{k}{2\pi}\int_{-2K}^{2K}\theta^*\,\mathrm{sn}\,\gamma^*\,\mathrm{d}\gamma^*+\frac{1}{2\pi}\int_{-2K}^{2K}\theta^* Z(\gamma^*)\,\mathrm{d}\gamma^*,$$

and
$$0 = \Omega_{hm}-\frac{k}{2\pi}\int_{-2K}^{2K}\theta^*\,\mathrm{sn}\,\gamma^*\,\mathrm{d}\gamma^*+\frac{1}{2\pi}\int_{-2K}^{2K}\theta^* Z(\gamma^*)\,\mathrm{d}\gamma^*.$$

These two equations give immediately that

$$\int_{-2K}^{2K}\theta^*\,\mathrm{sn}\,\gamma^*\,\mathrm{d}\gamma^* = 0, \tag{10}$$

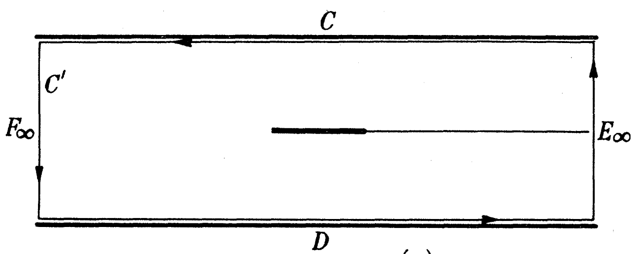
and
$$\Omega_{hm} = -\frac{1}{2\pi}\int_{-2K}^{2K}\theta^* Z(\gamma^*)\,\mathrm{d}\gamma^*. \tag{11}$$

4. The circulation

There exists a further relation, apart from (9), connecting the functions θ^* and X. This is in the form of an integral equation, and may be found by considering an expression for the circulation about the aerofoil and wake.

The circulation is defined to be
$$\Gamma \equiv \int_{C'} \mathbf{q}.\mathrm{d}\mathbf{s},$$

where d**s** represents an element of length in the physical plane, and C' is any closed contour enclosing both the aerofoil and the wake. The circuit selected is illustrated in figure 3 (a), representing the z plane, and is readily seen to correspond to the line EFE in the t plane, as shown in figure 3 (b).



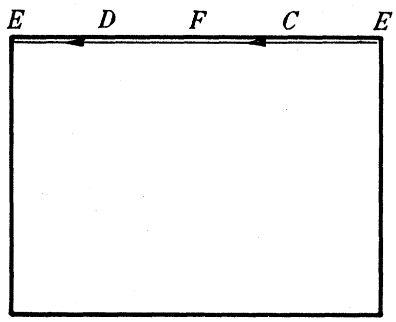


Figure 3. (a) The z plane; (b) the t plane.

The theory to follow is valid only as an approximation of first order in the perturbation velocity $q-U$. However, this approximation is a consequence of the assumption that the displacements of the aerofoil from its mean steady position are of small amplitude, and hence does not introduce any additional errors. To first order in this perturbation velocity, Ω is directly proportional to the velocity q. For

$$\Omega \equiv \ln\left(\frac{U}{q}\right) = -\ln\left(1+\frac{q}{U}-1\right) \approx -\left(\frac{q}{U}-1\right), \quad \text{to first order.}$$

Thus
$$q \approx U(1-\Omega).$$

On the tunnel walls, we have that $\mathrm{d}s = \mathrm{d}x$. Also, differentiation of equation (1) gives

$$-\mathrm{sn}\, t\, \mathrm{dn}\, t \mathrm{d}t = -\frac{\pi k'}{2hk}\cosh\frac{\pi z}{2h}\mathrm{d}z.$$

But
$$\cosh\frac{\pi z}{2h} = \sqrt{\left(1+\sinh^2\frac{\pi z}{2h}\right)} = \sqrt{\left(1+\frac{k^2}{k'^2}\mathrm{cn}^2\, t\right)} = \frac{1}{k'}\mathrm{dn}\, t.$$

Hence
$$\operatorname{sn} t \, dt = (\pi/2hk)\, dz. \tag{12}$$
Therefore on the walls, that is, on $z = x \pm ih$, or $t = \gamma + iK'$, (12) gives
$$\operatorname{ns}\gamma\, d\gamma = (\pi/2h)\, dx. \tag{13}$$
With the aid of these results the circulation can now be written
$$\begin{aligned} \Gamma &= \frac{2hU}{\pi}\int_{2K}^{-2K}[1-\Omega(\gamma+iK')]\operatorname{ns}\gamma\, d\gamma \\ &= \frac{2hU}{\pi}\int_{-2K}^{2K}\Omega(\gamma+iK')\operatorname{ns}\gamma\, d\gamma, \end{aligned} \tag{14}$$
where $\int_{-2K}^{2K}\operatorname{ns}\gamma\, d\gamma = 0$, since $\operatorname{ns}\gamma$ is an odd function of γ.

Subsequent calculations are much simplified if $f(t)$ is expressed in yet another form. Using (9) to (11) we find (see appendix 1)
$$f(t) = \frac{1}{2\pi}\int_{-2K}^{2K}\theta^*\frac{\operatorname{cn}\gamma^*\operatorname{dn}\gamma^*+\operatorname{cn} t\operatorname{dn} t}{\operatorname{sn}\gamma^*-\operatorname{sn} t}\, d\gamma^* + \frac{1}{2\pi}\int_0^{K'} X\operatorname{sn} t\frac{\operatorname{cn} t\operatorname{dn} t-\operatorname{cn} i\eta^*\operatorname{dn} i\eta^*}{\operatorname{sn}^2 i\eta^*-\operatorname{sn}^2 t}\, d\eta^*. \tag{15}$$
The periodic properties
$$\operatorname{sn}(\gamma+iK') = \frac{1}{k}\operatorname{ns}\gamma, \quad \operatorname{cn}(\gamma+iK') = -\frac{i\operatorname{dn}\gamma}{k\operatorname{sn}\gamma}, \quad \operatorname{dn}(\gamma+iK') = -\frac{i\operatorname{cn}\gamma}{\operatorname{sn}\gamma},$$
give that on the walls
$$\begin{aligned} \Omega(\gamma+iK') = &\frac{1}{2\pi}\int_{-2K}^{2K}\theta^*\operatorname{ns}\gamma\frac{k\operatorname{cn}\gamma^*\operatorname{dn}\gamma^*\operatorname{sn}^2\gamma-\operatorname{cn}\gamma\operatorname{dn}\gamma}{k\operatorname{sn}\gamma^*\operatorname{sn}\gamma-1}\, d\gamma^* \\ &-\frac{1}{2\pi}\int_0^{K'} X\operatorname{ns}\gamma\frac{k\operatorname{cn} i\eta^*\operatorname{dn} i\eta^*\operatorname{sn}^2\gamma+\operatorname{cn}\gamma\operatorname{dn}\gamma}{k^2\operatorname{sn}^2 i\eta^*\operatorname{sn}^2\gamma-1}\, d\eta^*. \end{aligned}$$
Substitution of this into equation (14) now yields for the circulation
$$\begin{aligned} \Gamma = &\frac{hU}{\pi^2}\int_{-2K}^{2K}\theta^*\int_{-2K}^{2K}\operatorname{ns}^2\gamma\frac{k\operatorname{cn}\gamma^*\operatorname{dn}\gamma^*\operatorname{sn}^2\gamma-\operatorname{cn}\gamma\operatorname{dn}\gamma}{k\operatorname{sn}\gamma^*\operatorname{sn}\gamma-1}\, d\gamma\, d\gamma^* \\ &-\frac{hU}{\pi^2}\int_0^{K'} X\int_{-2K}^{2K}\operatorname{ns}^2\gamma\frac{k\operatorname{cn} i\eta^*\operatorname{dn} i\eta^*\operatorname{sn}^2\gamma+\operatorname{cn}\gamma\operatorname{dn}\gamma}{k^2\operatorname{sn}^2 i\eta^*\operatorname{sn}^2\gamma-1}\, d\gamma\, d\eta^*. \end{aligned}$$
In appendix 2 it is shown that
$$I_1 \equiv \int_{-2K}^{2K}\operatorname{ns}^2\gamma\frac{k\operatorname{cn}\gamma^*\operatorname{dn}\gamma^*\operatorname{sn}^2\gamma-\operatorname{cn}\gamma\operatorname{dn}\gamma}{k\operatorname{sn}\gamma^*\operatorname{sn}\gamma-1}\, d\gamma = -4Kk[\operatorname{cn}\gamma^*\operatorname{dn}\gamma^*+\operatorname{sn}\gamma^* Z(\gamma^*)],$$
and
$$I_2 \equiv \int_{-2K}^{2K}\operatorname{ns}^2\gamma\frac{k\operatorname{cn} i\eta^*\operatorname{dn} i\eta^*\operatorname{sn}^2\gamma+\operatorname{cn}\gamma\operatorname{dn}\gamma}{k^2\operatorname{sn}^2 i\eta^*\operatorname{sn}^2\gamma-1}\, d\gamma = -4Kk[\operatorname{cn} i\eta^*\operatorname{dn} i\eta^*+\operatorname{sn} i\eta^* Z(i\eta^*)].$$
Hence, using these results,
$$\begin{aligned} \Gamma = -\frac{4KhkU}{\pi^2}\Big\{&\int_{-2K}^{2K}\theta^*[\operatorname{cn}\gamma^*\operatorname{dn}\gamma^*+\operatorname{sn}\gamma^* Z(\gamma^*)]\, d\gamma^* \\ &-\int_0^{K'} X[\operatorname{cn} i\eta^*\operatorname{dn} i\eta^*+\operatorname{sn} i\eta^* Z(i\eta^*)]\, d\eta^*\Big\}. \end{aligned} \tag{16}$$

We now assume that the circulation about the aerofoil and wake prior to the start of the unsteady motion was zero. Then by Kelvin's circulation theorem, it is zero throughout the unsteady motion. That is,

$$\int_{-2K}^{2K} \theta^*[\operatorname{cn}\gamma^*\,\operatorname{dn}\gamma^*+\operatorname{sn}\gamma^* Z(\gamma^*)]\,d\gamma^*-\int_0^{K'} X[\operatorname{cn}i\eta^*\,\operatorname{dn}i\eta^*+\operatorname{sn}i\eta^* Z(i\eta^*)]\,d\eta^*=0. \tag{17}$$

This is the integral equation relating X and the flow direction θ^*.

5. Strength of the vortex sheet

The function X, the jump in Ω across the wake, is to first order in the perturbation velocity $q-U$ directly proportional to the strength of the vortex sheet. For if we denote by $[A]$ the jump in the quantity A across the wake, then the strength of the vortex sheet is defined as

$$[q].$$

Since we have, to first order, that $\Omega=-(q/U-1)$, as shown in § 4, therefore

$$X\equiv[\Omega]=-[q/U-1];$$

that is,

$$X=-[q]/U, \tag{18}$$

directly proportional to the strength of the sheet.

The function X may be determined from the condition (9), the integral equation (17) and a differential equation obtainable from the wake condition, namely, that the pressure is continuous across the vortex sheet.

Bernoulli's equation may be written

$$p+\tfrac{1}{2}\rho q^2+\rho\frac{\partial\phi}{\partial t}=C, \tag{19}$$

where p is the pressure, ρ the density and C a function of time only. Suppose that s is distance measured along the vortex sheet, then on each side of the sheet it follows that

$$\frac{\partial p}{\partial s}+\rho q\frac{\partial q}{\partial s}+\rho\frac{\partial q}{\partial t}=0,$$

since $q=\partial\phi/\partial s$. As the vortex sheet lies on $y=0$, $2a\leqslant x<\infty$, we put $s=x$, and so

$$\frac{\partial p}{\partial x}+\rho q\frac{\partial q}{\partial x}+\rho\frac{\partial q}{\partial t}=0.$$

Subtraction of the two equations appropriate to the opposite sides of the sheet yields

$$\frac{\partial}{\partial x}[\tfrac{1}{2}q^2]+\frac{\partial}{\partial t}[q]=0,$$

since the pressure is continuous across the wake, that is, $[p]=0$. Using the results

$$q\approx U(1-\Omega),\quad q^2\approx U^2(1-2\Omega)$$

and (18) we may write the last equation as

$$U\frac{\partial X}{\partial x}+\frac{\partial X}{\partial t}=0. \tag{20}$$

On the wake $t = \pm 2K + \mathrm{i}\eta$, and so equation (1) becomes

$$\operatorname{cn} \mathrm{i}\eta = -\frac{k'}{k} \sinh \frac{\pi x}{2h}. \tag{21}$$

Hence, by a method used previously,

$$\operatorname{sn} \mathrm{i}\eta \, \mathrm{d}\eta = \frac{\mathrm{i}\pi}{2hk} \mathrm{d}x, \tag{22}$$

which allows (20) to be written

$$\frac{\pi U}{2hk} \mathrm{i} \operatorname{ns} \mathrm{i}\eta \frac{\partial \mathrm{X}}{\partial \eta} + \frac{\partial \mathrm{X}}{\partial t} = 0. \tag{23}$$

This is the differential equation satisfied by X.

The flow direction on the aerofoil surface, θ^*, is not completely known *a priori*, since it is composed not only of a term arising from the prescribed upwash distribution, but also of a term associated with the motion of the front stagnation point. This latter term is not prescribed, and must be calculated.

Suppose θ^* is divided into the two components mentioned. If the upwash velocity on the plate is v, then, except in the neighbourhood of the front stagnation point, $\theta^* = \sin^{-1}(v/q)$. That is, to first order in the perturbation velocity, $\theta^* = v/U$.

The position of the front stagnation point in the mean steady flow is $\gamma^* = 0$ in the t plane. Assume that at any instant in the unsteady motion the front stagnation point is at $\gamma^* = -\delta$, where δ will be small since the amplitude of the unsteady motion is small. Then in the range $-\delta \leqslant \gamma^* \leqslant 0$, the flow is reversed, that is, θ^* is increased by π in this range. Thus we may write

$$\theta^* = \begin{cases} v/U & (-2K \leqslant \gamma^* \leqslant 2K), \\ \pi & (-\delta \leqslant \gamma^* \leqslant 0). \end{cases} \tag{24}$$

Substituting from (24) into equations (9), (10) and (17) we obtain, to first order of magnitude in δ,

$$\int_{-2K}^{2K} v \operatorname{sn} \gamma \, \mathrm{d}\gamma = 0, \tag{25}$$

$$\pi\delta + \frac{1}{U} \int_{-2K}^{2K} v \, \mathrm{d}\gamma + \int_0^{K'} \mathrm{X} \, \mathrm{d}\eta = 0, \tag{26}$$

and $$\pi\delta + \frac{1}{U} \int_{-2K}^{2K} v [\operatorname{cn} \gamma \operatorname{dn} \gamma + \operatorname{sn} \gamma Z(\gamma)] \, \mathrm{d}\gamma - \int_0^{K'} \mathrm{X} [\operatorname{cn} \mathrm{i}\eta \operatorname{dn} \mathrm{i}\eta + \operatorname{sn} \mathrm{i}\eta Z(\mathrm{i}\eta)] \, \mathrm{d}\eta = 0. \tag{27}$$

The upwash v is prescribed, and hence equations (26) and (27) enable δ and X to be determined for any general upwash distribution.

6. The pressure distribution

An expression is now derived for the pressure distribution on the aerofoil surface. The lift and moment, as will be seen subsequently, are functions of the pressure.

Bernoulli's equation at the aerofoil surface can be written

$$p = \tfrac{1}{2}\rho(U^2 - q^2) - \rho \frac{\partial}{\partial t} \int q \, \mathrm{d}s + A, \tag{28}$$

where A is a function of time only, and $q = \partial\phi/\partial s$. Writing $s = x$ on the aerofoil surface, we obtain from equation (12),

$$\mathrm{d}s = \mathrm{d}x = (2hk/\pi) \operatorname{sn} \gamma \, \mathrm{d}\gamma.$$

Replacing q by $U(1-\Omega)$, as previously, we have that

$$\frac{\partial q}{\partial t} = -U\frac{\partial \Omega}{\partial t}, \quad U^2 - q^2 = 2U^2\Omega.$$

Therefore (28) can be written

$$p = \rho U^2 \Omega + \frac{2hk\rho U}{\pi}\int_0^{\gamma} \operatorname{sn}\gamma \frac{\partial \Omega}{\partial t}\,\mathrm{d}\gamma, \tag{29}$$

the constant being omitted since its contribution to the lift and moment would be zero.

Equation (15), in the limit as $\eta \to 0$, gives on the aerofoil surface,

$$\Omega(\gamma) = \frac{1}{2\pi}\int_{-2K}^{2K} \theta^* \frac{\operatorname{cn}\gamma^* \operatorname{dn}\gamma^* + \operatorname{cn}\gamma \operatorname{dn}\gamma}{\operatorname{sn}\gamma^* - \operatorname{sn}\gamma}\,\mathrm{d}\gamma^* + \frac{1}{2\pi}\int_0^{K'} \mathrm{X} \operatorname{sn}\gamma \frac{\operatorname{cn}\gamma \operatorname{dn}\gamma - \operatorname{cn} \mathrm{i}\eta^* \operatorname{dn} \mathrm{i}\eta^*}{\operatorname{sn}^2 \mathrm{i}\eta^* - \operatorname{sn}^2\gamma}\,\mathrm{d}\eta^*. \tag{30}$$

It is shown in appendix 3 that substitution of Ω from (30) into (29) eventually yields a comparatively simple expression for p. The pressure is found to be

$$p = \frac{\rho U^2}{2\pi}\int_{-2K}^{2K} \theta^* \left[\frac{\operatorname{cn}\gamma^* \operatorname{dn}\gamma^* + \operatorname{cn}\gamma \operatorname{dn}\gamma}{\operatorname{sn}\gamma^* - \operatorname{sn}\gamma} - Z(\gamma)\right]\mathrm{d}\gamma^*$$
$$+ \frac{hk\rho U}{\pi^2}\int_{-2K}^{2K} \dot{\theta}^* \int_0^{\gamma} \operatorname{sn}\gamma^* \left[\frac{\operatorname{cn}\gamma^* \operatorname{dn}\gamma^* + \operatorname{cn}\gamma \operatorname{dn}\gamma}{\operatorname{sn}\gamma^* - \operatorname{sn}\gamma} + Z(\gamma^*)\right]\mathrm{d}\gamma\,\mathrm{d}\gamma^*, \tag{31}$$

where $\dot{\theta}^* \equiv \partial\theta^*/\partial t$. Equation (31) reveals the important fact that the pressure distribution, and hence also the lift and moment, can be obtained without direct calculation of X, the strength of the vortex sheet.

The above formula for p may yet be simplified considerably. Let

$$G(\gamma^*) \equiv \frac{2hk}{\pi}\int_0^{\gamma^*} \theta^* \operatorname{sn}\gamma^* \mathrm{d}\gamma^*. \tag{32}$$

Then it is shown in appendix 3 that (31) can be written

$$p = \frac{\rho U}{2\pi}\int_{-2K}^{2K} (U\theta^* + \dot{G}^*)\left[\frac{\operatorname{cn}\gamma^* \operatorname{dn}\gamma^* + \operatorname{cn}\gamma \operatorname{dn}\gamma}{\operatorname{sn}\gamma^* - \operatorname{sn}\gamma} - Z(\gamma)\right]\mathrm{d}\gamma^*. \tag{33}$$

Replacing θ^* in (32) and (33) by its components from equation (24), we obtain to first order in δ,

$$p = -\tfrac{1}{2}\rho U^2 \delta \frac{1 + \operatorname{cn}\gamma \operatorname{dn}\gamma + \operatorname{sn}\gamma Z(\gamma)}{\operatorname{sn}\gamma} + \frac{\rho U}{2\pi}\int_{-2K}^{2K} (v + \dot{g})\left[\frac{\operatorname{cn}\gamma^* \operatorname{dn}\gamma^* + \operatorname{cn}\gamma \operatorname{dn}\gamma}{\operatorname{sn}\gamma^* - \operatorname{sn}\gamma} - Z(\gamma)\right]\mathrm{d}\gamma^*, \tag{34}$$

where
$$g(\gamma^*) \equiv \frac{2hk}{\pi U}\int_0^{\gamma^*} v \operatorname{sn}\gamma^* \,\mathrm{d}\gamma^*. \tag{35}$$

Equation (34) gives the pressure distribution for an arbitrary upwash v. It may be compared in form with the corresponding result for an unsteady aerofoil in a free jet (Woods 1955 *b*).

7. The lift and moment

Exact analytical expressions are now obtained for the lift and moment acting on the aerofoil. The lift, L, and the moment about the mid-chord point, M, are given by

$$\left.\begin{aligned} L &= -\int_{\gamma=-2K}^{2K} p\,\mathrm{d}x(\gamma), \\ M &= \int_{\gamma=-2K}^{2K} xp\,\mathrm{d}x(\gamma). \end{aligned}\right\} \tag{36}$$

With the aid of (12), the lift may be written

$$L = -\frac{2hk}{\pi}\int_{-2K}^{2K} p\,\mathrm{sn}\,\gamma\,\mathrm{d}\gamma,$$

and so substituting for p from (34), we have

$$L = \frac{hk\rho U^2}{\pi}\left\{\delta\int_{-2K}^{2K}[1+\mathrm{cn}\,\gamma\;\mathrm{dn}\,\gamma+\mathrm{sn}\,\gamma Z(\gamma)]\,\mathrm{d}\gamma - \frac{1}{\pi U}\int_{-2K}^{2K}\int_{-2K}^{2K}(v+\dot{g})\left[\mathrm{sn}\,\gamma\frac{\mathrm{cn}\,\gamma^*\;\mathrm{dn}\,\gamma^*+\mathrm{cn}\,\gamma\;\mathrm{dn}\,\gamma}{\mathrm{sn}\,\gamma^*-\mathrm{sn}\,\gamma}-\mathrm{sn}\,\gamma Z(\gamma)\right]\mathrm{d}\gamma\,\mathrm{d}\gamma^*\right\} \tag{37}$$

The integrals of equation (37) may be readily evaluated (see appendix 2), and we eventually obtain for the lift

$$L = \frac{4Khk\rho U^2}{\pi}\left\{\delta+\frac{1}{\pi U}\int_{-2K}^{2K}(v+\dot{g})\,[\mathrm{cn}\,\gamma^*\;\mathrm{dn}\,\gamma^*+\mathrm{sn}\,\gamma^* Z(\gamma^*)]\,\mathrm{d}\gamma^*\right\}. \tag{38}$$

An expression for the moment may be obtained similarly. From equation (12),

$$\mathrm{sn}\,\gamma\,\mathrm{d}\gamma = \frac{\pi}{2hk}\mathrm{d}x.$$

Hence

$$x = \frac{2hk}{\pi}\int_0^{\gamma}\mathrm{sn}\,\gamma\,\mathrm{d}\gamma = \frac{h}{\pi}\ln\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right). \tag{39}$$

Substitution from (39) into (36) therefore gives

$$M = 2k\left(\frac{h}{\pi}\right)^2\int_{-2K}^{2K} p\,\mathrm{sn}\,\gamma\ln\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)\mathrm{d}\gamma, \tag{40}$$

and so from (34) the moment about the mid-chord point is

$$M = -k\rho\left(\frac{hU}{\pi}\right)^2\left\{\delta\int_{-2K}^{2K}[1+\mathrm{cn}\,\gamma\;\mathrm{dn}\,\gamma+\mathrm{sn}\,\gamma Z(\gamma)]\ln\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)\mathrm{d}\gamma - \frac{1}{\pi U}\int_{-2K}^{2K}\int_{-2K}^{2K}(v+\dot{g})\left[\mathrm{sn}\,\gamma\frac{\mathrm{cn}\,\gamma^*\;\mathrm{dn}\,\gamma^*+\mathrm{cn}\,\gamma\;\mathrm{dn}\,\gamma}{\mathrm{sn}\,\gamma^*-\mathrm{sn}\,\gamma}-\mathrm{sn}\,\gamma Z(\gamma)\right]\ln\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)\mathrm{d}\gamma\,\mathrm{d}\gamma^*\right\}. \tag{41}$$

The integrals occurring here are calculated in appendix 2, yielding finally

$$M = -k\rho\left(\frac{hU}{\pi}\right)^2\left\{\delta\left[\frac{8(E-K)}{k}+I\right]-\frac{1}{\pi U}\int_{-2K}^{2K}(v+\dot{g})\,[I'(\gamma^*)-I]\,\mathrm{d}\gamma^*\right\}, \tag{42}$$

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where E is the complete elliptic integral of the second kind,

$$E \equiv \int_0^K \mathrm{dn}^2 u \, \mathrm{d}u$$

(see, for example, Byrd & Friedman 1954, p. 191),

$$I'(\gamma^*) \equiv \int_{-2K}^{2K} \frac{\mathrm{sn}\,\gamma\,\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma}{\mathrm{sn}\,\gamma^* - \mathrm{sn}\,\gamma} \ln\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right) \mathrm{d}\gamma, \tag{43}$$

and

$$I \equiv \int_{-2K}^{2K} \mathrm{sn}\,\gamma Z(\gamma) \ln\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right) \mathrm{d}\gamma = \frac{1}{4K}\int_{-2K}^{2K} I'(u)\,\mathrm{d}u. \tag{44}$$

Equations (38) and (42) give the general analytical solutions for the lift and moment for an arbitrary upwash distribution.

8. Forces due to a sudden upgust

The above general expressions for the lift and moment for an arbitrary upwash are now examined for particular types of unsteady motion. The first case is where the upwash is an upgust impulsively imposed on a steady motion.

It was previously stated that the equations (20), (26) and (27) completely determine X and δ for any general upwash distribution. We now proceed to investigate these functions for the case of a sudden upgust.

Equation (20) indicates that X is a function of x and t of the form

$$\mathrm{X} = \mathrm{X}(x - Ut). \tag{45}$$

This shows that the value of X at a general point (x, t) is equal to its value at some other point (x_0, t_0), say, where $t_0 = t-(x-x_0)/U$. In particular, the value of X at (x, t) is equal to that at the trailing edge $x = 2a$ at a 'retarded time' $t-(x-2a)/U$. That is,

$$\mathrm{X}(x, Ut) = \mathrm{X}(2a, Ut - x + 2a).$$

By defining new variables

$$\xi = x/2a, \quad \tau = Ut/2a, \tag{46}$$

we may write this last result as

$$\mathrm{X}(\xi, \tau) = \mathrm{X}(1, \tau - \xi + 1). \tag{47}$$

The variable of the integrals involving X in equations (26) and (27) is now changed from η to ξ. From equations (1), (2) and (22) we find

$$\left.\begin{aligned} \mathrm{d}\eta &= \frac{\pi}{\sqrt{(2hk)}} \frac{\mathrm{d}x}{\sqrt{\left\{\cosh\dfrac{\pi x}{h} - \dfrac{1+k^2}{1-k^2}\right\}}}, \quad \mathrm{cn}\,\mathrm{i}\eta = \frac{k'}{k}\sinh\frac{\pi x}{2h}, \quad \mathrm{dn}\,\mathrm{i}\eta = k'\cosh\frac{\pi x}{2h}, \\ \mathrm{i}\eta &= \mathrm{cn}^{-1}\left[\frac{k'}{k}\sinh\frac{\pi x}{2h}\right], \quad \mathrm{sn}\,\mathrm{i}\eta\,\mathrm{d}\eta = \frac{\pi\mathrm{i}\,\mathrm{d}x}{2hk}. \end{aligned}\right\} \tag{48}$$

Thus, substitution from (46) and (48) enables us to write

$$\left.\begin{aligned} &\int_0^{K'} \mathrm{X}\,\mathrm{d}\eta = \frac{r}{\sqrt{2k'}}\int_1^\infty \frac{\mathrm{X}\,\mathrm{d}\xi}{\sqrt{(\cosh r\xi - \cosh r)}}, \\ \text{and}\quad &\int_0^{K'} \mathrm{X}[\mathrm{cn}\,\mathrm{i}\eta\,\mathrm{dn}\,\mathrm{i}\eta + \mathrm{sn}\,\mathrm{i}\eta Z(\mathrm{i}\eta)]\,\mathrm{d}\eta \\ &\quad = \frac{rk'}{2\sqrt{2}\,k}\int_1^\infty \frac{\mathrm{X}\sinh r\xi\,\mathrm{d}\xi}{\sqrt{(\cosh r\xi - \cosh r)}} + \frac{\mathrm{i}r}{2k}\int_1^\infty \mathrm{X}Z\left\{\mathrm{cn}^{-1}\left(\frac{k'}{k}\sinh\frac{r\xi}{2}\right)\right\}\mathrm{d}\xi, \end{aligned}\right\} \tag{49}$$

where

$$r = 2\pi a/h.$$

Next suppose the unsteady motion to have been in progress for a finite time, and let ξ_s be the value of ξ at the end of the (finite) vortex sheet. Then from equation (47),

$$\mathrm{X}(\xi_s, \tau) = \mathrm{X}(1, \tau - \xi_s + 1).$$

This shows that the unsteady motion may be regarded as having commenced at the retarded time $\tau - \xi_s + 1$, which may therefore conveniently be taken as the origin in the time scale. That is, we put $\xi_s = 1 + \tau$, so that $\mathrm{X}(\xi, \tau) = 0$ for $\xi > 1 + \tau$. Then the substitution

$$\nu = \tau - \xi + 1$$

and equations (49) allow (26) and (27) to be written in the form

$$\pi\delta + \frac{1}{U}\int_{-2K}^{2K} v \, \mathrm{d}\gamma + \frac{r}{\sqrt{2k'}}\int_0^\tau \frac{\mathrm{X}(1,\nu)\,\mathrm{d}\nu}{\sqrt{\{\cosh r(\tau-\nu+1) - \cosh r\}}} = 0 \tag{50}$$

and
$$\pi\delta + \frac{1}{U}\int_{-2K}^{2K} v[\mathrm{cn}\,\gamma\;\mathrm{dn}\,\gamma + \mathrm{sn}\,\gamma Z(\gamma)]\,\mathrm{d}\gamma - \frac{rk'}{2\sqrt{2k}}\int_0^\tau \frac{\mathrm{X}(1,\nu)\sinh r(\tau-\nu+1)\,\mathrm{d}\nu}{\sqrt{\{\cosh r(\tau-\nu+1)-\cosh r\}}}$$
$$- \frac{\mathrm{i}r}{2k}\int_0^\tau \mathrm{X}(1,\nu)\, Z\Big\{\mathrm{cn}^{-1}\Big(\frac{k'}{k}\sinh\frac{r}{2}[\tau-\nu+1]\Big)\Big\}\,\mathrm{d}\nu = 0. \tag{51}$$

Equations (50) and (51) are integral equations with 'difference' kernels, and hence solutions can be obtained using the Laplace transform method. The Laplace transform notation used is

$$\mathscr{L}\{f(\tau), p\} \equiv \int_0^\infty \mathrm{e}^{-p\tau} f(\tau)\,\mathrm{d}\tau \equiv \bar{f}(p), \tag{52}$$

where p is a parameter whose real part is greater than some constant, c_0 say, which is just large enough to ensure that the integral converges. The Faltung theorem of the Laplace transform theory, which is needed, is

$$\mathscr{L}\Big\{\int_0^\tau F(\sigma)\,\Phi(\tau-\sigma)\,\mathrm{d}\sigma, p\Big\} = \mathscr{L}\{F(\tau),p\}\mathscr{L}\{\Phi(\tau),p\}. \tag{53}$$

Taking the transform of (50) and (51), and using (53), we obtain

$$\pi(\bar{\delta} + \bar{a}) + \frac{r}{\sqrt{2k'}}\bar{\mathrm{X}}\int_0^\infty \frac{\mathrm{e}^{-p\tau}\,\mathrm{d}\tau}{\sqrt{\{\cosh r(\tau+1) - \cosh r\}}} = 0 \tag{54}$$

and
$$\pi(\bar{\delta} + \bar{b}) - \frac{rk'}{2\sqrt{2k}}\bar{\mathrm{X}}\int_0^\infty \frac{\mathrm{e}^{-p\tau}\sinh r(\tau+1)\,\mathrm{d}\tau}{\sqrt{\{\cosh r(\tau+1)-\cosh r\}}}$$
$$-\frac{\mathrm{i}r}{2k}\bar{\mathrm{X}}\int_0^\infty \mathrm{e}^{-p\tau} Z\Big\{\mathrm{cn}^{-1}\Big(\frac{k'}{k}\sinh\frac{r}{2}[\tau+1]\Big)\Big\}\,\mathrm{d}\tau = 0, \tag{55}$$

where
$$a \equiv \frac{1}{\pi U}\int_{-2K}^{2K} v\,\mathrm{d}\gamma, \tag{56}$$

and
$$b \equiv \frac{1}{\pi U}\int_{-2K}^{2K} v[\mathrm{cn}\,\gamma\;\mathrm{dn}\,\gamma + \mathrm{sn}\,\gamma Z(\gamma)]\,\mathrm{d}\gamma. \tag{57}$$

Consider now the integral

$$J \equiv \int_0^\infty \mathrm{e}^{-p\tau}\left\{\frac{\frac{k'}{\sqrt{2}}\sinh r(\tau+1)}{\sqrt{\{\cosh r(\tau+1)-\cosh r\}}} + \mathrm{i}Z\Big[\mathrm{cn}^{-1}\Big(\frac{k'}{k}\sinh\frac{r}{2}\{\tau+1\}\Big)\Big]\right\}\mathrm{d}\tau,$$

which occurs in equation (55). This may be written

$$J=\int_0^\infty e^{-p\tau}\left\{\frac{\frac{k'}{\sqrt{2}}\sinh r(\tau+1)}{\sqrt{\{\cosh r(\tau+1)-\cosh r\}}}+iZ\left[\mathrm{cn}^{-1}\left(\frac{k'}{k}\sinh\frac{r}{2}\{\tau+1\}\right)\right]-\frac{\frac{\sqrt{2}k}{k'}}{\sqrt{\{\cosh r(\tau+1)-\cosh r\}}}\right\}d\tau+\frac{\sqrt{2}k}{k'}\int_0^\infty\frac{e^{-p\tau}\,d\tau}{\sqrt{\{\cosh r(\tau+1)-\cosh r\}}}.$$

We now integrate by parts the integral

$$I\equiv\int_0^\infty e^{-p\tau}\left\{\frac{\frac{k'}{\sqrt{2}}\sinh r(\tau+1)}{\sqrt{\{\cosh r(\tau+1)-\cosh r\}}}+iZ\left[\mathrm{cn}^{-1}\left(\frac{k'}{k}\sinh\frac{r}{2}\{\tau+1\}\right)\right]-\frac{\frac{\sqrt{2}k}{k'}}{\sqrt{\{\cosh r(\tau+1)-\cosh r\}}}\right\}d\tau.$$

The addition and subtraction of a term, in the manner indicated, is necessary in order that the integrated part should vanish and the resultant integral should be convergent. Thus we find, on integrating by parts,

$$I=\frac{1}{p}\int_0^\infty e^{-p\tau}\frac{\partial}{\partial\tau}\left\{\frac{\frac{k'}{\sqrt{2}}\sinh r(\tau+1)}{\sqrt{\{\cosh r(\tau+1)-\cosh r\}}}+iZ\left[\mathrm{cn}^{-1}\left(\frac{k'}{k}\sinh\frac{r}{2}\{\tau+1\}\right)\right]-\frac{\frac{\sqrt{2}k}{k'}}{\sqrt{\{\cosh r(\tau+1)-\cosh r\}}}\right\}d\tau.$$

From the definition of $Z(u)$, and the results

$$\frac{d}{du}(\mathrm{cn}^{-1}u)=-\frac{1}{\sqrt{\{(1-u^2)(k'^2+k^2u^2)\}}}\quad\text{and}\quad \mathrm{dn}^2(\mathrm{cn}^{-1}u)=k'^2+k^2u^2$$

(see, for example, Byrd & Friedman 1954, pp. 31, 285), we obtain, after some rearrangement,

$$I=\frac{rk'}{2\sqrt{2}p}\left\{\int_0^\infty\frac{e^{-p\tau}(1-\cosh r\tau)\,d\tau}{[\cosh r(\tau+1)-\cosh r]^{\frac{3}{2}}}+\left(\frac{2E}{Kk'^2}-1\right)\int_0^\infty\frac{e^{-p\tau}\,d\tau}{\sqrt{\{\cosh r(\tau+1)-\cosh r\}}}\right\}.$$

With this result, equation (55) can now be written

$$\pi(\bar{\delta}+\bar{b})-\frac{r}{2k}\int_0^\infty\left\{\frac{rk'}{2\sqrt{2}p}\frac{(1-\cosh r\tau)}{[\cosh r(\tau+1)-\cosh r]}+\frac{rk'}{2\sqrt{2}p}\left(\frac{2E}{Kk'^2}-1\right)+\frac{\sqrt{2}k}{k'}\right\}\times\frac{e^{-p\tau}\,d\tau}{\sqrt{\{\cosh r(\tau+1)-\cosh r\}}}=0. \tag{58}$$

The integrals of equations (54) and (58) may be evaluated in terms of Legendre functions of the second kind. From Erdelyi (1953, p. 155) we have the definition of the associated Legendre function of the second kind:

$$Q_\nu^\mu(\cosh r)=(\tfrac{1}{2}\pi)^{\frac{1}{2}}e^{\mu\pi i}\frac{(\sinh r)^\mu}{\Gamma(\frac{1}{2}-\mu)}\int_r^\infty e^{-(\nu+\frac{1}{2})u}(\cosh u-\cosh r)^{-\mu-\frac{1}{2}}\,du, \tag{59}$$

(where Γ is the gamma function), provided

$$\mathscr{R}(\nu+\mu+1)>0,\quad \mathscr{R}\mu<\tfrac{1}{2}. \tag{60}$$

After an appropriate change of variable, it can easily be shown that both the integrals under consideration satisfy the requirement (60). Therefore we find, after some use of standard recurrence relations,

$$\left.\begin{aligned}&\int_0^\infty \frac{e^{-p\tau}\,d\tau}{\surd\{\cosh r(\tau+1)-\cosh r\}} = \frac{\surd 2\, e^{p}}{r} Q_{p/r-\frac{1}{2}}(\cosh r),\\ J=&\int_0^\infty\left\{\frac{rk'}{2\surd 2p}\frac{(1-\cosh r\tau)}{[\cosh r(\tau+1)-\cosh r]}+\frac{rk'}{2\surd 2p}\left(\frac{2E}{Kk'^2}-1\right)+\frac{\surd 2k}{k'}\right\}\frac{e^{-p\tau}\,d\tau}{\surd\{\cosh r(\tau+1)-\cosh r\}}\\ &=\frac{e^{p}k'}{2p}\left\{\left(\frac{p}{r}-\frac{1}{2}\right)Q_{p/r-\frac{3}{2}}(\cosh r)+\left(\frac{2E}{Kk'^2}-1\right)Q_{p/r-\frac{1}{2}}(\cosh r)-\left(\frac{p}{r}+\frac{1}{2}\right)Q_{p/r+\frac{1}{2}}(\cosh r)\right\}.\end{aligned}\right\} \quad (61)$$

If we define quantities N and $\mathscr{Q}$ by

$$N\equiv\frac{p}{r}-\frac{1}{2},$$

and $$\mathscr{Q}(\cosh r)\equiv Q_{N+1}(\cosh r)+Q_{N-1}(\cosh r)+2\left(1-\frac{2E}{Kk'^2}\right)Q_N(\cosh r),$$

then equations (54) and (58) may now be written respectively,

$$\pi(\bar\delta+\bar a)+\overline{X}\frac{e^{p}}{k'}Q_N(\cosh r)=0, \quad (62)$$

and $$\pi(\bar\delta+\bar b)+\overline{X}\frac{e^{p}k'}{4k}\left[Q_{N+1}(\cosh r)-Q_{N-1}(\cosh r)+\frac{r}{2p}\mathscr{Q}(\cosh r)\right]=0. \quad (63)$$

Recurrence relations for the Legendre functions (Whittaker & Watson 1946, p. 318) enable us to write

$$Q_{N+1}(\cosh r)-Q_{N-1}(\cosh r)=\frac{2N+1}{N(N+1)}\sinh r Q'_N(\cosh r),$$

where $Q'_N(\cosh r)$ is the associated Legendre function; that is, using equation (2) and the fact that $r=2\pi a/h$,

$$Q_{N+1}(\cosh r)-Q_{N-1}(\cosh r)=\frac{2(2N+1)}{N(N+1)}\frac{k}{k'^2}Q'_N(\cosh r).$$

Thus (63) may alternatively be expressed as

$$\pi(\bar\delta+\bar b)+\overline{X}\frac{e^{p}}{k'}\left\{\mathscr{N}Q'_N(\cosh r)+\frac{rk'^2}{8pk}\mathscr{Q}(\cosh r)\right\}=0, \quad (64)$$

where $\mathscr{N}\equiv(2N+1)/2N(N+1)$.

Solving equations (62) and (64) for $\overline{X}$ and $\bar\delta$, we obtain

$$\overline{X}=\frac{\pi k'(\bar b-\bar a)\,e^{-p}}{Q_N-\mathscr{N}Q'_N-\dfrac{rk'^2}{8pk}\mathscr{Q}}, \quad (65)$$

and $$\bar\delta=\frac{(\bar a-\bar b)\,Q_N}{Q_N-\mathscr{N}Q'_N-\dfrac{rk'^2}{8pk}\mathscr{Q}}-\bar a, \quad (66)$$

where we have put Q for $Q(\cosh r)$ and $\mathscr{Q}$ for $\mathscr{Q}(\cosh r)$. The latter equation may be written

$$\bar\delta=\bar\delta_1 p\bar g-\bar a, \quad (67)$$

where
$$\bar{\delta}_1 p \equiv \frac{Q_N}{Q_N - \mathscr{N} Q'_N - \dfrac{rk'^2}{8pk}\mathscr{Q}}, \tag{68}$$
and
$$g \equiv a - b$$
$$= \frac{1}{\pi U}\int_{-2K}^{2K} v[1 - \operatorname{cn}\gamma \operatorname{dn}\gamma - \operatorname{sn}\gamma Z(\gamma)]\,\mathrm{d}\gamma, \tag{69}$$
from (56) and (57). The Laplace transform relation
$$\mathscr{L}\left\{\frac{\mathrm{d}F}{\mathrm{d}\tau}, p\right\} = p\mathscr{L}\{F(\tau), p\} - F(0), \tag{70}$$
where $F(0)$ is the value of F at $\tau = 0$, together with equation (57), enables (67) to be written
$$\bar{\delta} = \mathscr{L}\{\delta_1, p\}[\mathscr{L}\{g', p\} + g(0)] - \bar{a}.$$
Thus, using the Faltung theorem,
$$\delta(\tau) = \int_0^\tau \delta_1(\tau - \sigma)\,\mathrm{d}g(\sigma) - a. \tag{71}$$
Finally, the inversion integral of the Laplace transform theory, namely,
$$\mathscr{L}^{-1}\{\bar{f}(p), \tau\} = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^{p\tau}\bar{f}(p)\,\mathrm{d}p = f(\tau) \quad (c > c_0), \tag{72}$$
and the definition of $\bar{\delta}_1$ given in equation (68) give that
$$\delta_1 = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{\mathrm{e}^{p\tau}Q_N}{Q_N - \mathscr{N}Q'_N - \dfrac{rk'^2}{8pk}\mathscr{Q}}\frac{\mathrm{d}p}{p}. \tag{73}$$
Equations (67), (71) and (73) fix the value of δ. Hence the expression for the lift and moment may be written down once the functional form of the upgust is given. A formula for X, though not required in the determination of the lift and moment, may be obtained by a similar procedure to the above.

9. The harmonically oscillating aerofoil

The second case to be considered, and one which is of common practical interest, is that of an aerofoil performing harmonic oscillations. Formulae are obtained on the assumption that the upwash velocity is varying harmonically with time. These formulae can be obtained as a special case of the results of § 8, taking the upwash in the above equations to have persisted for an infinitely long time. Alternatively, as is done here, they may be found directly.

Suppose the upwash to have frequency $\nu/2\pi$. Then a solution of equation (20) is
$$\mathrm{X} = \mathrm{X}_0\,\mathrm{e}^{\mathrm{i}\nu(t - x/U)},$$
where X_0 is a constant, while the velocity v may be written
$$v(\gamma, t) = v_0(\gamma)\,\mathrm{e}^{\mathrm{i}\nu t}.$$
The variable of equations (26) and (27) is again changed from η to ξ, where we now put
$$\frac{x}{2a} = 1 + \xi, \quad \lambda = \frac{2a\nu}{U}, \quad r = \frac{2\pi a}{h}. \tag{74}$$

Then with the aid of (48), equations (26) and (27) become

$$\pi\delta + \frac{e^{i\nu t}}{U}\int_{-2K}^{2K} v_0\,d\gamma + \frac{X_0\,e^{i\nu t}\,r\,e^{-i\lambda}}{\sqrt{2k'}}\int_0^\infty \frac{e^{-i\lambda\xi}\,d\xi}{\sqrt{\{\cosh r(\xi+1)-\cosh r\}}} = 0, \tag{75}$$

and

$$\pi\delta + \frac{e^{i\nu t}}{U}\int_{-2K}^{2K} v_0[\text{cn}\,\gamma\;\text{dn}\,\gamma + \text{sn}\,\gamma Z(\gamma)]\,d\gamma - \frac{X_0\,e^{i\nu t}\,r\,e^{-i\lambda}}{2k}\left\{\frac{k'}{\sqrt{2}}\int_0^\infty \frac{e^{-i\lambda\xi}\sinh r(\xi+1)\,d\xi}{\sqrt{\{\cosh r(\xi+1)-\cosh r\}}}\right.$$
$$\left. + i\int_0^\infty e^{-i\lambda\xi} Z\left[\text{cn}^{-1}\left(\frac{k'}{k}\sinh\frac{r}{2}\{\xi+1\}\right)\right]d\xi\right\} = 0. \tag{76}$$

It is immediately obvious that the integrals of equations (75) and (76) are identical with those of (54) and (55), with p replaced by $i\lambda$. Solutions are again obtainable in terms of Legendre functions, using the formulae of equation (61). We find for equations (75) and (76)

$$\pi\delta + \frac{e^{i\nu t}}{U}\int_{-2K}^{2K} v_0\,d\gamma + \frac{X_0\,e^{i\nu t}}{k'}\,Q_N(\cosh r) = 0, \tag{77}$$

and

$$\pi\delta + \frac{e^{i\nu t}}{U}\int_{-2K}^{2K} v_0[\text{cn}\,\gamma\;\text{dn}\,\gamma + \text{sn}\,\gamma Z(\gamma)]\,d\gamma + \frac{X_0\,e^{i\nu t}}{k'}\left\{\mathcal{N} Q'_N(\cosh r) + \frac{rk'^2}{8i\lambda k}\mathcal{Q}(\cosh r)\right\} = 0, \tag{78}$$

where $N \equiv i\lambda/r - \frac{1}{2}$, and, as before, $\mathcal{N} = (2N+1)/2N(N+1)$.

Solving equations (77) and (78) for X_0 and δ, we find

$$X_0 = \frac{-\dfrac{k'}{U}\displaystyle\int_{-2K}^{2K} v_0[1-\text{cn}\,\gamma\;\text{dn}\,\gamma - \text{sn}\,\gamma Z(\gamma)]\,d\gamma}{Q_N - \mathcal{N} Q'_N - \dfrac{rk'^2}{8i\lambda k}\mathcal{Q}}, \tag{79}$$

and $$\delta = -\frac{e^{i\nu t}}{2\pi U}\left\{\int_{-2K}^{2K} v_0[1+\text{cn}\,\gamma\;\text{dn}\,\gamma + \text{sn}\,\gamma Z(\gamma)]\,d\gamma - \frac{Q_N + \mathcal{N} Q'_N + \dfrac{rk'^2}{8i\lambda k}\mathcal{Q}}{Q_N - \mathcal{N} Q'_N - \dfrac{rk'^2}{8i\lambda k}\mathcal{Q}}\right.$$
$$\left.\times\int_{-2K}^{2K} v_0[1-\text{cn}\,\gamma\;\text{dn}\,\gamma - \text{sn}\,\gamma Z(\gamma)]\,d\gamma\right\}. \tag{80}$$

If for brevity we define $$a_0 \equiv \frac{1}{\pi U}\int_{-2K}^{2K} v_0\,d\gamma, \tag{81}$$

$$b_0 \equiv \frac{1}{\pi U}\int_{-2K}^{2K} v_0[\text{cn}\,\gamma\;\text{dn}\,\gamma + \text{sn}\,\gamma Z(\gamma)]\,d\gamma, \tag{82}$$

and $$T(\lambda, r) \equiv -\frac{Q_N + \mathcal{N} Q'_N + \dfrac{rk'^2}{8i\lambda k}\mathcal{Q}}{Q_N - \mathcal{N} Q'_N - \dfrac{rk'^2}{8i\lambda k}\mathcal{Q}}, \tag{83}$$

then equation (80) may be written,

$$\delta = -e^{i\nu t}\{\tfrac{1}{2}a_0(1+T) + \tfrac{1}{2}b_0(1-T)\}. \tag{84}$$

It will be shown subsequently that the function $T(\lambda, r)$ is a generalization of the well-known T function of Küssner (1940) for an oscillating aerofoil in an infinite stream.

The lift and moment on the aerofoil for a harmonic upwash may now be expressed as

$$L = -\frac{4Khk\rho U^2\,\mathrm{e}^{\mathrm{i}\nu t}}{\pi}\left\{a_0\frac{1+T}{2}+b_0\frac{1-T}{2}-\frac{1}{\pi U}\int_{-2K}^{2K}(v_0+\dot{g}_0)\,[\mathrm{cn}\,\gamma^*\;\mathrm{dn}\,\gamma^*+\mathrm{sn}\,\gamma^* Z(\gamma^*)]\,\mathrm{d}\gamma^*\right\}, \tag{85}$$

and

$$M = k\rho\left(\frac{hU}{\pi}\right)^2\mathrm{e}^{\mathrm{i}\nu t}\left\{\left[a_0\frac{1+T}{2}+b_0\frac{1-T}{2}\right]\left[\frac{8(E-K)}{k}+I\right]+\frac{1}{\pi U}\int_{-2K}^{2K}(v_0+\dot{g}_0)\,[I'(\gamma^*)-I]\,\mathrm{d}\gamma^*\right\}, \tag{86}$$

where here
$$\dot{g}_0 = \frac{2\mathrm{i}\nu hk}{\pi U}\int_0^{\gamma^*} v_0\,\mathrm{sn}\,\gamma^*\,\mathrm{d}\gamma^*. \tag{87}$$

10. Motion of a rigid aerofoil

The solutions (85) and (86) for a harmonic upwash are most frequently applied in practice to an aerofoil regarded as a rigid body. Two types of oscillations of interest are: (i) translational motion of the aerofoil, that is, pitching, and (ii) simple rotation of the aerofoil about some fixed point, here taken to be the mid-chord point. Substitution of the appropriate form for v_0 in equations (81) to (87) will yield the required formulae for the two cases respectively.

Instead of considering these two motions separately, we take them simultaneously as a compound oscillation, and extract the components after the general results have been obtained. Let the translational and rotational displacements be given by

$$y = y^0\,\mathrm{e}^{\mathrm{i}\nu t}, \quad \alpha = \alpha^0\,\mathrm{e}^{\mathrm{i}\nu t},$$

respectively, where y^0, α^0 are constants. Then the vertical displacement corresponding to a rotation α may be expressed as

$$-y' = -x\alpha = -x\alpha^0\,\mathrm{e}^{\mathrm{i}\nu t}.$$

Therefore the upwash velocity for the compound motion is

$$v = \frac{\mathrm{d}}{\mathrm{d}t}(y-y') = \frac{\mathrm{d}y}{\mathrm{d}t}-\left(\frac{\partial y'}{\partial t}+\frac{\partial y'}{\partial x}\frac{\partial x}{\partial t}\right) = \mathrm{e}^{\mathrm{i}\nu t}\{\mathrm{i}\nu(y^0-x\alpha^0)-\alpha^0 U\}. \tag{88}$$

Since (39) gives
$$x = \frac{h}{\pi}\ln\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right),$$

the velocity may therefore be expressed as

$$v_0 = \mathrm{i}\nu\left[y^0-\frac{h\alpha^0}{\pi}\ln\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)\right]-\alpha^0 U. \tag{89}$$

Substituting (89) into equations (81), (82) and (87), we find that

$$a_0 = \frac{4K}{\pi U}(\mathrm{i}\nu y^0-\alpha^0 U), \tag{90}$$

$$b_0 = -\frac{\mathrm{i}\nu h\alpha^0}{\pi^2 U}\left[\frac{8(E-K)}{k}+I\right], \tag{91}$$

and $$\dot{g}_0 = \frac{i\nu h}{\pi U}(i\nu y^0 - \alpha^0 U)\ln\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right) + \frac{\nu^2 h^2 \alpha^0}{2\pi^2 U}\left[\ln\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)\right]^2. \tag{92}$$

Using these expressions for a_0, b_0 and $\dot{g}_0$ in equations (85) and (86), we obtain for the lift, L, and moment about the mid-chord point, M,

$$L = -\frac{4Khk\rho U\,\mathrm{e}^{i\nu t}}{\pi^2}\left\{2K(1+T)(i\nu y^0 - \alpha^0 U) - \frac{i\nu h\alpha^0}{2\pi}(1-T)\left[\frac{8(E-K)}{k}+I\right]\right.$$
$$\left. - \frac{i\nu h}{\pi U}(i\nu y^0 - 2\alpha^0 U)\left[\frac{8(E-K)}{k}+I\right]\right\}, \tag{93}$$

and $$M = k\rho\left(\frac{hU}{\pi}\right)^2 \frac{\mathrm{e}^{i\nu t}}{\pi U}\left\{2K(1+T)(i\nu y^0 - \alpha^0 U)\left[\frac{8(E-K)}{k}+I\right]\right.$$
$$\left. - \frac{i\nu h\alpha^0}{2\pi}(1-T)\left[\frac{8(E-K)}{k}+I\right]^2 + \frac{\nu^2 h^2 \alpha^0}{2\pi^2 U}(J'-J)\right\}, \tag{94}$$

where $$J' \equiv \int_{-2K}^{2K} I'(\gamma)\left[\ln\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)\right]^2 \mathrm{d}\gamma, \tag{95}$$

and $$J \equiv \int_{-2K}^{2K} I\left[\ln\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)\right]^2 \mathrm{d}\gamma. \tag{96}$$

In practical investigations of the unsteady motion of an aerofoil, both in a free stream and a wind tunnel, the aerodynamic forces are usually discussed in terms of certain non-dimensional quantities, derived from the lift and moment, called 'air-load' coefficients. Following Jones (1941), we introduce dimensionless numbers l_{12}, l_{34}, m_{12}, m_{34}, defined by

$$\frac{L}{2\pi a\rho U^2\,\mathrm{e}^{i\nu t}} = l_{12}\frac{y^0}{2a} + l_{34}\alpha^0, \tag{97}$$

and $$\frac{M}{4\pi a^2 \rho U^2\,\mathrm{e}^{i\nu t}} = m_{12}\frac{y^0}{2a} + m_{34}\alpha^0. \tag{98}$$

Then the air-load coefficients are defined to be the real and imaginary parts of these numbers, that is, l_1, l_2, etc., where $l_1 + il_2 = l_{12}$, etc.

Substituting from (93) and (94) we now obtain

$$l_{12} = -\frac{4Khk}{\pi^3 U}\left\{2K(1+T)\,i\nu + \frac{\nu^2 h}{\pi U}\left[\frac{8(E-K)}{k}+I\right]\right\}, \tag{99}$$

$$l_{34} = \frac{2Khk}{\pi^3 aU}\left\{2KU(1+T) + \frac{i\nu h}{2\pi}(1-T)\left[\frac{8(E-K)}{k}+I\right] - \frac{2i\nu h}{\pi}\left[\frac{8(E-K)}{k}+I\right]\right\}, \tag{100}$$

$$m_{12} = \frac{i\nu Kh^2 k}{\pi^4 aU}\left\{(1+T)\left[\frac{8(E-K)}{k}+I\right]\right\}, \tag{101}$$

and

$$m_{34} = -\frac{h^2 k}{4\pi^4 a^2 U}\left\{2KU(1+T)\left[\frac{8(E-K)}{k}+I\right] + \frac{i\nu h}{2\pi}(1-T)\left[\frac{8(E-K)}{k}+I\right]^2 - \frac{\nu^2 h^2}{2\pi^2 U}(J'-J)\right\}. \tag{102}$$

Thus the air-load coefficients are found to be

$$\left.\begin{aligned}
l_1 &= \frac{4Khkv}{\pi^3 U}\left\{2KU\mathscr{I}(T)-\frac{vh}{\pi U}\left[\frac{8(E-K)}{k}+I\right]\right\},\\
l_2 &= -\frac{8K^2hkv}{\pi^3 U}[1+\mathscr{R}(T)],\\
l_3 &= \frac{2Khk}{\pi^3 aU}\left\{2KU[1+\mathscr{R}(T)]+\frac{vh}{2\pi}\mathscr{I}(T)\left[\frac{8(E-K)}{k}+I\right]\right\},\\
l_4 &= \frac{2Khk}{\pi^3 aU}\left\{2KU\mathscr{I}(T)-\frac{vh}{2\pi}[3+\mathscr{R}(T)]\left[\frac{8(E-K)}{k}+I\right]\right\},\\
m_1 &= -\frac{Kh^2kv}{\pi^4 aU}\mathscr{I}(T)\left[\frac{8(E-K)}{k}+I\right],\\
m_2 &= \frac{Kh^2kv}{\pi^4 aU}[1+\mathscr{R}(T)]\left[\frac{8(E-K)}{k}+I\right],\\
m_3 &= -\frac{h^2k}{4\pi^4a^2U}\left\{2KU[1+\mathscr{R}(T)]\left[\frac{8(E-K)}{k}+I\right]+\frac{vh}{2\pi}\mathscr{I}(T)\left[\frac{8(E-K)}{k}+I\right]^2-\frac{v^2h^2}{2\pi^2U}(J'-J),\right.\\
m_4 &= -\frac{h^2k}{4\pi^4a^2U}\left[\frac{8(E-K)}{k}+I\right]\left\{2KU\mathscr{I}(T)+\frac{vh}{2\pi}[1-\mathscr{R}(T)]\left[\frac{8(E-K)}{k}+I\right]\right\}.
\end{aligned}\right\}\tag{103}$$

These expressions give exact formulae for the air-load coefficients as functions of the real and imaginary parts of the function $T(\lambda, r)$, as defined in equation (83).

11. Expansions in series

It remains now to evaluate the functions $I'(\gamma)$, I, J' and J occurring in the formulae of equation (103). To do this it is found necessary to expand in series the integrals by which these functions are defined, and to integrate to an appropriate number of terms. The expansions are made as power series in the elliptic parameter k since, as will be indicated below, this is the parameter involved in obtaining the first-order wind-tunnel corrections.

It is shown in appendix 2 that

$$I'(\gamma^*) \equiv \int_{-2K}^{2K} \frac{\operatorname{sn}\gamma \operatorname{cn}\gamma \operatorname{dn}\gamma}{\operatorname{sn}\gamma^* - \operatorname{sn}\gamma} \ln\left(\frac{1-k\operatorname{cd}\gamma}{1+k\operatorname{cd}\gamma}\right) d\gamma \tag{43}$$

$$= \frac{8K}{k}\left(\operatorname{dn}^2\gamma^* - \frac{E}{K}\right) - 8Kk\operatorname{sn}\gamma^* \int_0^{\gamma^*} \operatorname{sn}\gamma^* Z(\gamma^*)\, d\gamma^*. \tag{104}$$

Further, some useful expansions obtained from the Fourier series for the elliptic functions (see, for example, Byrd & Friedman, p. 303) are

$$\left.\begin{aligned}
\operatorname{sn}\gamma &= \sin\gamma - \tfrac{1}{4}k^2\cos\gamma(\gamma-\sin\gamma\cos\gamma)+O(k^4),\\
\operatorname{cn}\gamma &= \cos\gamma + \tfrac{1}{4}k^2\sin\gamma(\gamma-\sin\gamma\cos\gamma)+O(k^4),\\
\operatorname{dn}\gamma &= 1-\tfrac{1}{2}k^2\sin^2\gamma+O(k^4),
\end{aligned}\right\}\tag{105}$$

provided $|k|<1$. With the aid of these expansions, the definition

$$Z(\gamma) \equiv \int_0^\gamma \operatorname{dn}^2 u\, du - \frac{E}{K}\gamma,$$

and the approximations

$$E = \tfrac{1}{2}\pi[1-\tfrac{1}{4}k^2-\tfrac{3}{64}k^4+O(k^6)], \quad K = \tfrac{1}{2}\pi[1+\tfrac{1}{4}k^2+\tfrac{9}{64}k^4+O(k^6)] \tag{106}$$

(Byrd & Friedman, pp. 297, 298), we obtain

$$Z(\gamma) = \tfrac{1}{4}k^2 \sin 2\gamma + O(k^4). \tag{107}$$

Hence
$$\int_0^{\gamma^*} \operatorname{sn}\gamma^* Z(\gamma^*)\,\mathrm{d}\gamma^* = \tfrac{1}{4}k^2 \int_0^{\gamma^*} \sin\gamma^* \sin 2\gamma^*\,\mathrm{d}\gamma^* + O(k^4)$$
$$= \tfrac{1}{6}k^2 \sin^3\gamma^* + O(k^4) = \tfrac{1}{6}k^2 \operatorname{sn}^3\gamma^* + O(k^4),$$

the last step of which follows from (105). Substitution of this into (104) now gives

$$I'(\gamma^*) = \frac{8K}{k}\left(\operatorname{dn}^2\gamma^* - \frac{E}{K}\right) - \frac{4K}{3}k^3 \operatorname{sn}^4\gamma^* + O(k^5). \tag{108}$$

With this result equation (44) gives immediately that

$$I = -\tfrac{1}{4}\pi k^3 + O(k^5). \tag{109}$$

Evaluation of the quantities J' and J requires the expansion of the function

$$\left[\ln\left(\frac{1-k\operatorname{cd}\gamma}{1+k\operatorname{cd}\gamma}\right)\right]^2$$

in powers of k. We find that

$$\left[\ln\left(\frac{1-k\operatorname{cd}\gamma}{1+k\operatorname{cd}\gamma}\right)\right]^2 = 4k^2[\operatorname{cd}^2\gamma + \tfrac{2}{3}k^2\operatorname{cd}^4\gamma + O(k^4)],$$

so, by (95),

$$J' = 32Kk\int_{-2K}^{2K} \operatorname{cd}^2\gamma\left(\operatorname{dn}^2\gamma - \frac{E}{K}\right)\mathrm{d}\gamma + \tfrac{16}{3}Kk^3\int_{-2K}^{2K}\left\{4\operatorname{cd}^4\gamma\left(\operatorname{dn}^2\gamma - \frac{E}{K}\right) - k^2\operatorname{sn}^4\gamma\operatorname{cd}^2\gamma\right\}\mathrm{d}\gamma + O(k^7)$$
$$= \frac{128}{k}(E^2 - k'^2K^2) + \tfrac{14}{3}\pi K k^5 + O(k^7). \tag{110}$$

Similarly, from (96),
$$J = -\pi^2 k^5 + O(k^7). \tag{111}$$

The expansions (108) to (111) give approximate values of functions required in the determination of the lift, moment and air-load coefficients.

12. Limiting case of an infinite stream

In the limiting case of an infinite stream, that is, as the wall separation h becomes infinite, all the above results should reduce to the well-known free-stream formulae. From equations (1) and (2) we have that as $h \to \infty$,

$$k \to 0, \quad k' \to 1, \quad K \to \tfrac{1}{2}\pi, \quad K' \to \infty. \tag{112}$$

From (105), the elliptic functions are seen to degenerate to ordinary trigonometrical functions, namely,
$$\operatorname{sn} u \to \sin u, \quad \operatorname{cn} u \to \cos u, \quad \operatorname{dn} u \to 1, \tag{113}$$

while from (106) and (107) we find that

$$E \to \tfrac{1}{2}\pi, \quad Z(u) \to 0. \tag{114}$$

With the aid of these, the circulation given by equation (16) becomes

$$\Gamma = -2aU\left[\int_{-\pi}^{\pi} \theta^* \cos\gamma^*\,\mathrm{d}\gamma^* - \int_0^{\infty} \mathrm{X}\cosh\eta^*\,\mathrm{d}\eta^*\right], \tag{115}$$

which is an alternative form of a well-known result. The equations (9), (10) and (17) reduce to the classical free-stream boundary conditions

$$\int_{-\pi}^{\pi} \theta^* \,\mathrm{d}\gamma^* + \int_0^\infty \mathrm{X}\,\mathrm{d}\eta^* = 0, \tag{116}$$

$$\int_{-\pi}^{\pi} \theta^* \sin\gamma^* \,\mathrm{d}\gamma^* = 0, \tag{117}$$

and

$$\int_{-\pi}^{\pi} \theta^* \cos\gamma^* \,\mathrm{d}\gamma^* - \int_0^\infty \mathrm{X}\cosh\eta^* \,\mathrm{d}\eta^* = 0. \tag{118}$$

Formulae (115) to (118) are to be found in Woods (1954).

Similarly, equations (38) and (41) for the lift and moment degenerate, with the aid of (108) and (109), to (see, for example, Greidanus & van Heemert 1948)

$$L = 2\pi a\rho U^2\left\{\delta + \frac{1}{\pi U}\int_{-2K}^{2K} (v+\dot{g})\cos\gamma^*\,\mathrm{d}\gamma^*\right\}, \tag{119}$$

where now

$$g = \frac{2a}{U}\int_0^{\gamma^*} v\sin\gamma^*\,\mathrm{d}\gamma^*, \tag{120}$$

and

$$M = 2\pi a^2\rho U^2\left\{\delta + \frac{1}{\pi U}\int_{-\pi}^{\pi} (v+\dot{g})\cos 2\gamma^*\,\mathrm{d}\gamma^*\right\}. \tag{121}$$

The reduction of the formulae in the special cases of §§ 8 to 10 requires the limiting values of the Legendre functions. These may be obtained from the formula (Watson 1952, p. 156)

$$\lim_{n=\infty}\left[\frac{n^{-m}\sin n\pi}{\sin(n+m)\pi}\,Q_n^m\left(\cosh\frac{z}{n}\right)\right] = K_m(z),$$

where $K_m(z)$ is a modified Bessel function of the second kind. From equation (74),

$$r = 2\pi a/h \to 0 \quad \text{as} \quad h\to\infty,$$

and so

$$N = \mathrm{i}\lambda/r - \tfrac{1}{2} \to \infty \quad \text{as} \quad h\to\infty.$$

Also $\mathscr{N} = (2N+1)/2N(N+1) = O(1/N)$. Thus putting $z = \mathrm{i}\lambda - \frac{1}{2}r$, $n = N = \mathrm{i}\lambda/r - \frac{1}{2}$, and $m = 0, 1$, in the above formula, we find that

$$\left.\begin{aligned} \lim_{r=0} Q_N(\cosh r) &= K_0(\mathrm{i}\lambda), \\ \lim_{r=0} \mathscr{N} Q_N'(\cosh r) &= -K_1(\mathrm{i}\lambda), \\ \lim_{r=0} \mathscr{Q}(\cosh r) &= 0. \end{aligned}\right\} \tag{122}$$

Using these limiting values, the function $T(\lambda, r)$ defined in the case of harmonic oscillations reduces to

$$T(\lambda, 0) = T_0 = \frac{K_1(\mathrm{i}\lambda) - K_0(\mathrm{i}\lambda)}{K_1(\mathrm{i}\lambda) + K_0(\mathrm{i}\lambda)},$$

or, in terms of Hankel functions,

$$T_0 = \frac{H_1^{(2)}(\lambda) - \mathrm{i}H_0^{(2)}(\lambda)}{H_1^{(2)}(\lambda) + \mathrm{i}H_0^{(2)}(\lambda)}. \tag{123}$$

This is the well-known Küssner free-stream function $T(\lambda)$.

The limiting values of the air-load coefficients can now be obtained. In place of the frequency ν, we use the non-dimensional 'reduced frequency' of the oscillation, λ, defined in equation (74). Since the aerofoil has zero thickness, we may take $4a = c$, where c is the length of the aerofoil chord. Thus, from (74),

$$\lambda = \frac{2a\nu}{U} = \frac{c\nu}{2U}. \tag{124}$$

Using equations (106), (109) to (111) and (123), we find for the limiting values of the non-dimensional numbers of equations (99) to (106),

$$\left.\begin{aligned} l_{12} &= -\mathrm{i}\lambda(1+T_0)+\lambda^2, \\ l_{34} &= (1+T_0)-\tfrac{1}{2}\mathrm{i}\lambda(1-T_0)+2\mathrm{i}\lambda, \\ m_{12} &= -\tfrac{1}{2}\mathrm{i}\lambda(1+T_0), \\ m_{34} &= \tfrac{1}{2}(1+T_0)-\tfrac{1}{4}\mathrm{i}\lambda(1-T_0)+\tfrac{1}{8}\lambda^2. \end{aligned}\right\} \tag{125}$$

These agree with the values quoted by Greidanus & van Heemert (1948).

13. First-order tunnel corrections

The air-load coefficients defined above differ from their corresponding free-stream values by amounts which depend on the wall separation h. Equation (2) shows that the elliptic parameter k is inversely proportional to h; further, it is directly proportional to the ratio of aerofoil length to tunnel width. Thus an expansion of the formulae for the air-load coefficients as power series in k^2 allows the corrections necessary for reduction to free stream values to be readily evaluated. These series, which are based on the expansions of § 11, are taken to order k^2 only, and hence the corrections are first-order terms. The expansions below are easily shown to be rapidly convergent, so that the first-order terms give a good approximation to the corrections required.

From equation (2),

$$h = \frac{\pi a}{k}[1-\tfrac{1}{3}k^2+O(k^4)], \tag{126}$$

and so using this, and equations (106) and (109) to (111), we may write the air-load coefficients as

$$\left.\begin{aligned} l_1 &= \lambda\mathscr{I}(T)+\lambda^2+\tfrac{1}{12}k^2\lambda[2\mathscr{I}(T)+\lambda]+O(k^4), \\ l_2 &= -\lambda[1+\mathscr{R}(T)]-\tfrac{1}{6}k^2\lambda[1+\mathscr{R}(T)]+O(k^4), \\ l_3 &= 1+\mathscr{R}(T)-\tfrac{1}{2}\lambda\mathscr{I}(T)+\tfrac{1}{12}k^2[2+2\mathscr{R}(T)-\tfrac{1}{2}\lambda\mathscr{I}(T)]+O(k^4), \\ l_4 &= \mathscr{I}(T)+\tfrac{1}{2}\lambda[3+\mathscr{R}(T)]+\tfrac{1}{12}k^2[2\mathscr{I}(T)+\tfrac{1}{2}\lambda\{3+\mathscr{R}(T)\}]+O(k^4), \\ m_1 &= \tfrac{1}{2}\lambda\mathscr{I}(T)+\tfrac{1}{24}k^2\lambda\mathscr{I}(T)+O(k^4), \\ m_2 &= -\tfrac{1}{2}\lambda[1+\mathscr{R}(T)]-\tfrac{1}{24}k^2\lambda[1+\mathscr{R}(T)]+O(k^4), \\ m_3 &= \tfrac{1}{2}[1+\mathscr{R}(T)]-\tfrac{1}{4}\lambda\mathscr{I}(T)+\tfrac{1}{8}\lambda^2+\tfrac{1}{24}k^2[1+\mathscr{R}(T)]+O(k^4), \\ m_4 &= \tfrac{1}{2}\mathscr{I}(T)-\tfrac{1}{4}\lambda[1-\mathscr{R}(T)]+\tfrac{1}{24}k^2\mathscr{I}(T)+O(k^4). \end{aligned}\right\} \tag{127}$$

Equations (127) contain no expansion of the function $T(\lambda, r)$. From its definition, a series for $T(\lambda, r)$ of the form

$$T(\lambda, r) = T_0(\lambda)+k^2T_1(\lambda)+k^4T_2(\lambda)+\ldots,$$

(where $r = 2\pi a/h = 2\tanh^{-1}k$) can be obtained. Some analysis shows, however, that this expansion is too slowly convergent, even for small values of k, to allow its use in (127). Hence it is most convenient to evaluate the function $T(\lambda, r)$ directly, without resort to an expansion of the above type. This may be done by transforming the Legendre functions of $T(\lambda, r)$ into hypergeometric functions, and then calculating the latter from their expansions in series. It is shown in appendix 4 that we may write

$$T(\lambda,r)=\frac{\dfrac{i\lambda}{r}F\left[\dfrac{i\lambda}{r}-\dfrac{1}{2},\dfrac{1}{2};\dfrac{i\lambda}{r};e^{-2r}\right]-\dfrac{\left(\dfrac{i\lambda}{r}+\dfrac{1}{2}\right)^2}{\left(\dfrac{i\lambda}{r}+1\right)}e^{-2r}F\left[\dfrac{i\lambda}{r}+\dfrac{3}{2},\dfrac{1}{2};\dfrac{i\lambda}{r}+2;e^{-2r}\right]+e^{-r}\left(\dfrac{2E}{Kk'^2}-1-\dfrac{4i\lambda k}{rk'^2}\right)F\left[\dfrac{i\lambda}{r}+\dfrac{1}{2},\dfrac{1}{2};\dfrac{i\lambda}{r}+1;e^{-2r}\right]}{\dfrac{i\lambda}{r}F\left[\dfrac{i\lambda}{r}-\dfrac{1}{2},\dfrac{1}{2};\dfrac{i\lambda}{r};e^{-2r}\right]-\dfrac{\left(\dfrac{i\lambda}{r}+\dfrac{1}{2}\right)^2}{\left(\dfrac{i\lambda}{r}+1\right)}e^{-2r}F\left[\dfrac{i\lambda}{r}+\dfrac{3}{2},\dfrac{1}{2};\dfrac{i\lambda}{r}+2;e^{-2r}\right]+e^{-r}\left(\dfrac{2E}{Kk'^2}-1+\dfrac{4i\lambda k}{rk'^2}\right)F\left[\dfrac{i\lambda}{r}+\dfrac{1}{2},\dfrac{1}{2};\dfrac{i\lambda}{r}+1;e^{-2r}\right]}, \quad (128)$$

where $F(a, b; c; u)$ is the hypergeometric function. The formula (128) is in agreement with the expression obtained by Timman (1951).

The first-order correction terms for the air-load coefficients are now obtained by subtracting from the real and imaginary parts of the numbers in equation (125) their counterparts in (127).

14. Comparison with Timman's results

It is of interest to compare the formulae of equation (127) with corresponding results of Timman (1951) and Lilley (1952). For the dimensionless number l_{12}, Timman obtains

$$l_{12}=-i\lambda\frac{8hq^{\frac{1}{2}}}{\pi a}\left[T+\frac{1-T}{2}\frac{\pi^2 q^{\frac{1}{2}}}{K^2 k}\sum_{m=0}^{\infty}q^m\frac{1+q^{2m+1}}{(1-q^{2m+1})^2}\right]\sum_{m=0}^{\infty}q^m\frac{1+q^{2m+1}}{(1-q^{2m+1})^2}+\lambda^2\frac{16h^2q}{\pi^2a^2}\sum_{n=0}^{\infty}\frac{q^{2n}}{2n+1}\frac{1+q^{2n+1}}{(1-q^{2n+1})^3},$$

where $q \equiv e^{-\pi K'/K}$. With the aid of equations (106) and (126), together with the expansion

$$q = \tfrac{1}{16}k^2[1+\tfrac{1}{8}k^2+O(k^4)]$$

(Byrd & Friedman 1954, p. 299), this reduces to

$$l_1 = \lambda\mathscr{I}(T)+\lambda^2+\tfrac{1}{24}k^2\lambda[\mathscr{I}(T)-7\lambda]+O(k^4),$$

and

$$l_2 = -\lambda[1+\mathscr{R}(T)]+\tfrac{1}{24}k^2\lambda[2-\mathscr{R}(T)]+O(k^4),$$

which differ from the results of equation (127). It may similarly be shown that the other air-load coefficients due to Timman are at variance with those of (127). The formulae of Lilley (1952) are identical with those of Timman, except that the former has used the function $C(\lambda, r)$, defined by $C \equiv \frac{1}{2}(1+T)$, in place of $T(\lambda, r)$.

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The methods of Timman and Lilley depend on operations involving Fourier series. The validity of some of these operations seems doubtful, and it is probable that this accounts for the discrepancies. For example, Timman obtains the 'formal' expansion

$$\frac{1-\operatorname{cn}\gamma\operatorname{dn}\gamma}{\operatorname{sn}\gamma}-Z(\gamma)\sim-\frac{\pi}{K}\sum_{n=1}^{\infty}(-1)^n\frac{1+q^n}{1-q^n}\sin n\psi.$$

After multiplying this by the series

$$\operatorname{sn}\gamma=\frac{2\pi}{Kk}\sum_{n=0}^{\infty}\frac{q^{n+\frac{1}{2}}\sin(2n+1)\psi}{1-q^{2n+1}},$$

and integrating, Timman finds that

$$\int_0^{2K}[\operatorname{cn}\gamma\operatorname{dn}\gamma+\operatorname{sn}\gamma Z(\gamma)]\,\mathrm{d}\gamma=2K-\frac{2\pi^2}{Kk}\sum_{m=0}^{\infty}q^{m+\frac{1}{2}}\frac{1+q^{2m+1}}{(1-q^{2m+1})^2},$$

which result is used in the derivation of all his formulae for the air-load coefficients. It is shown in appendix 2 of the present paper, however, that the above integral is equal to zero. This would indicate that the operations carried out on the formal series of the above type are not justified.

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Appendix 1. Forms of the function $f(t)$

The solution (7) of Laplace's equation derived by Woods (1955*a*) can be transformed into an alternative form by using the properties of the theta functions given in Whittaker & Watson (1946, chap. 21).

From Landen's transformation we have

$$\vartheta_1(u\,|\,\tau_1)\,\vartheta_2(u\,|\,\tau_1)=\text{const.}\times\vartheta_1(2u\,|\,\tau),$$

where $\tau=2\tau_1=\mathrm{i}K'/K$. Taking logarithms and differentiating,

$$\frac{\vartheta_1'}{\vartheta_1}(u\,|\,\tau_1)+\frac{\vartheta_2'}{\vartheta_2}(u\,|\,\tau_1)=2\frac{\vartheta_1'}{\vartheta_1}(2u\,|\,\tau)=2\mathrm{i}+2\frac{\vartheta_4'}{\vartheta_4}(2u+\tfrac{1}{2}\pi\tau\,|\,\tau)$$
$$=2\mathrm{i}+2\vartheta_3^2(0\,|\,\tau)\,Z[\vartheta_3^2(0\,|\,\tau)\,(2u+\tfrac{1}{2}\pi\tau)],\qquad(129)$$

from the definition of the zeta function. An equation involving theta functions is

$$\frac{\vartheta_4'}{\vartheta_4}(y)+\frac{\vartheta_4'}{\vartheta_4}(z)-\frac{\vartheta_4'}{\vartheta_4}(y+z) = \vartheta_2(0)\,\vartheta_3(0)\,\frac{\vartheta_1(y)\,\vartheta_1(z)\,\vartheta_1(y+z)}{\vartheta_4(y)\,\vartheta_4(z)\,\vartheta_4(y+z)}.$$

If we put $y = u+\frac{1}{2}\pi\tau$, $z = \frac{1}{2}\pi$, and use the periodic properties of the theta functions, this becomes

$$\frac{\vartheta_1'}{\vartheta_1}(u\,|\,\tau_1)-\frac{\vartheta_2'}{\vartheta_2}(u\,|\,\tau_1) = \vartheta_2^2(0\,|\,\tau_1)\,\frac{\vartheta_3(u\,|\,\tau_1)\,\vartheta_4(u\,|\,\tau_1)}{\vartheta_1(u\,|\,\tau_1)\,\vartheta_2(u\,|\,\tau_1)}$$

$$= \vartheta_2^2(0\,|\,\tau_1)\,\frac{\vartheta_3(0\,|\,\tau)}{\vartheta_2(0\,|\,\tau)}\,\mathrm{ns}\,[\vartheta_3^2(0\,|\,\tau)\,2u], \tag{130}$$

from Landen's transformation and the definition of the elliptic function. It is found that

$$\vartheta_3^2(0\,|\,\tau) = \frac{2K}{\pi}, \quad \vartheta_2^2(0\,|\,\tau_1)\,\frac{\vartheta_3(0\,|\,\tau)}{\vartheta_2(0\,|\,\tau)} = \frac{4K}{\pi};$$

further, the periodic properties of the zeta function give that

$$Z(u+\tau K) = Z(u+\mathrm{i}K') = Z(u)+\frac{\mathrm{cn}\,u\,\mathrm{dn}\,u}{\mathrm{sn}\,u}-\frac{\mathrm{i}\pi}{2K}.$$

Substituting these results into (129) and (130), we obtain

$$\frac{\vartheta_1'}{\vartheta_1}(u\,|\,\tau_1) = \frac{2K}{\pi}\left\{Z\left(\frac{4Ku}{\pi}\right)+\frac{\mathrm{cn}\,\frac{4Ku}{\pi}\,\mathrm{dn}\,\frac{4Ku}{\pi}+1}{\mathrm{sn}\,\frac{4Ku}{\pi}}\right\} \tag{131}$$

and

$$\frac{\vartheta_2'}{\vartheta_2}(u\,|\,\tau_1) = \frac{2K}{\pi}\left\{Z\left(\frac{4Ku}{\pi}\right)+\frac{\mathrm{cn}\,\frac{4Ku}{\pi}\,\mathrm{dn}\,\frac{4Ku}{\pi}-1}{\mathrm{sn}\,\frac{4Ku}{\pi}}\right\}. \tag{132}$$

Replacement of the theta functions of equation (7) from (131) and (132) leads immediately to the required alternative form (8).

A further alternative form may be obtained by using properties of elliptic and zeta functions (Whittaker & Watson 1946, chap. 22). With the aid of these properties it can be shown that

$$\frac{1+\mathrm{cn}\,(\sigma-\epsilon)\,\mathrm{dn}\,(\sigma-\epsilon)}{\mathrm{sn}\,(\sigma-\epsilon)}+Z(\sigma-\epsilon) = 2Z\tfrac{1}{2}(\sigma-\epsilon)+\frac{\mathrm{cn}\,\frac{1}{2}(\sigma-\epsilon)\,\mathrm{dn}\,\frac{1}{2}(\sigma-\epsilon)}{\mathrm{sn}\,\frac{1}{2}(\sigma-\epsilon)}.$$

Now let

$$D \equiv \frac{\mathrm{cn}\,\frac{1}{2}(\sigma-\epsilon)\,\mathrm{dn}\,\frac{1}{2}(\sigma-\epsilon)}{\mathrm{sn}\,\frac{1}{2}(\sigma-\epsilon)}-\frac{\mathrm{cn}\,\frac{1}{2}(\sigma+\epsilon)\,\mathrm{dn}\,\frac{1}{2}(\sigma+\epsilon)}{\mathrm{sn}\,\frac{1}{2}(\sigma+\epsilon)}+2\{Z\tfrac{1}{2}(\sigma-\epsilon)-Z\tfrac{1}{2}(\sigma+\epsilon)\}.$$

Then substituting new variables

$$u = \tfrac{1}{2}(\sigma+\epsilon), \quad v = \tfrac{1}{2}(\sigma-\epsilon),$$

we find, after rearranging, that

$$D = \frac{\mathrm{sn}\,u\,\mathrm{cn}\,v\,\mathrm{dn}\,v-\mathrm{sn}\,v\,\mathrm{cn}\,u\,\mathrm{dn}\,u}{\mathrm{sn}\,u\,\mathrm{sn}\,v}-2\{Z(u)-Z(v)\}.$$

Further,

$$\mathrm{sn}\,(u-v) = \frac{\mathrm{sn}\,u\,\mathrm{cn}\,v\,\mathrm{dn}\,v-\mathrm{sn}\,v\,\mathrm{cn}\,u\,\mathrm{dn}\,u}{1-k^2\,\mathrm{sn}^2\,u\,\mathrm{sn}^2\,v},$$

and

$$Z(u)-Z(v) = Z(u-v)-k^2\,\mathrm{sn}\,u\,\mathrm{sn}\,v\,\mathrm{sn}\,(u-v).$$

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Hence we obtain $D = \operatorname{sn}(u-v)\left[\dfrac{1}{\operatorname{sn} u \operatorname{sn} v} + k^2 \operatorname{sn} u \operatorname{sn} v\right] - 2Z(u-v).$

Results quoted by Byrd & Friedman (1954, p. 23) yield

$$\frac{1}{\operatorname{sn} u \operatorname{sn} v} = \frac{\operatorname{dn}(u-v)+\operatorname{dn}(u+v)}{\operatorname{cn}(u-v)-\operatorname{cn}(u+v)},$$

and

$$k^2 \operatorname{sn} u \operatorname{sn} v = \frac{\operatorname{dn}(u-v)-\operatorname{dn}(u+v)}{\operatorname{cn}(u-v)+\operatorname{cn}(u+v)}.$$

Therefore

$$D = \operatorname{sn}(u-v)\left[\frac{\operatorname{dn}(u-v)+\operatorname{dn}(u+v)}{\operatorname{cn}(u-v)-\operatorname{cn}(u+v)} + \frac{\operatorname{dn}(u-v)-\operatorname{dn}(u+v)}{\operatorname{cn}(u-v)+\operatorname{cn}(u+v)}\right] - 2Z(u-v)$$

$$= 2\operatorname{sn}\epsilon \frac{\operatorname{cn}\epsilon \operatorname{dn}\epsilon + \operatorname{cn}\sigma \operatorname{dn}\sigma}{\operatorname{sn}^2\sigma - \operatorname{sn}^2\epsilon} - 2Z(\epsilon), \tag{133}$$

on resubstituting the original variables and rearranging. Similarly it may be shown that

$$\frac{\operatorname{cn}\tfrac{1}{2}(\sigma-\epsilon)\operatorname{dn}\tfrac{1}{2}(\sigma-\epsilon)}{\operatorname{sn}\tfrac{1}{2}(\sigma-\epsilon)} + \frac{\operatorname{cn}\tfrac{1}{2}(\sigma+\epsilon)\operatorname{dn}\tfrac{1}{2}(\sigma+\epsilon)}{\operatorname{sn}\tfrac{1}{2}(\sigma+\epsilon)} + 2\{Z\tfrac{1}{2}(\sigma-\epsilon)+Z\tfrac{1}{2}(\sigma+\epsilon)\}$$
$$= 2\operatorname{sn}\sigma \frac{\operatorname{cn}\epsilon \operatorname{dn}\epsilon + \operatorname{cn}\sigma \operatorname{dn}\sigma}{\operatorname{sn}^2\sigma - \operatorname{sn}^2\epsilon} + 2Z(\sigma). \tag{134}$$

Hence by addition of (133) and (134),

$$\frac{\operatorname{cn}\tfrac{1}{2}(\sigma-\epsilon)\operatorname{dn}\tfrac{1}{2}(\sigma-\epsilon)}{\operatorname{sn}\tfrac{1}{2}(\sigma-\epsilon)} + 2Z\tfrac{1}{2}(\sigma-\epsilon) = \frac{1+\operatorname{cn}(\sigma-\epsilon)\operatorname{dn}(\sigma-\epsilon)}{\operatorname{sn}(\sigma-\epsilon)} + Z(\sigma-\epsilon)$$
$$= \frac{\operatorname{cn}\epsilon \operatorname{dn}\epsilon + \operatorname{cn}\sigma \operatorname{dn}\sigma}{\operatorname{sn}\sigma - \operatorname{sn}\epsilon} - Z(\epsilon) + Z(\sigma). \tag{135}$$

Using (135) in the first integral of equation (8), and (133) in the second integral, we obtain the desired form (15).

Appendix 2. Evaluation of integrals involving elliptic and zeta functions

In the expressions for the circulation (§ 4) the integrals

$$I_1 \equiv \int_{-2K}^{2K} \operatorname{ns}^2\gamma \frac{k \operatorname{cn}\gamma^* \operatorname{dn}\gamma^* \operatorname{sn}^2\gamma - \operatorname{cn}\gamma \operatorname{dn}\gamma}{k \operatorname{sn}\gamma^* \operatorname{sn}\gamma - 1} \mathrm{d}\gamma,$$

$$I_2 \equiv \int_{-2K}^{2K} \operatorname{ns}^2\gamma \frac{k \operatorname{cn} i\eta^* \operatorname{dn} i\eta^* \operatorname{sn}^2\gamma + \operatorname{cn}\gamma \operatorname{dn}\gamma}{k^2 \operatorname{sn}^2 i\eta^* \operatorname{sn}^2\gamma - 1} \mathrm{d}\gamma,$$

have to be calculated. Consider

$$I_1 = \int_{-2K}^{2K} \operatorname{ns}^2\gamma \frac{k \operatorname{cn}\gamma^* \operatorname{dn}\gamma^* \operatorname{sn}^2\gamma - \operatorname{cn}\gamma \operatorname{dn}\gamma}{k^2 \operatorname{sn}^2\gamma^* \operatorname{sn}^2\gamma - 1} \mathrm{d}\gamma.$$

Now

$$\int_{-2K}^{2K} \frac{\operatorname{ns}^2\gamma \operatorname{cn}\gamma \operatorname{dn}\gamma \,\mathrm{d}\gamma}{k^2 \operatorname{sn}^2\gamma^* \operatorname{sn}^2\gamma - 1} = 2\int_0^{2K} \frac{\operatorname{ns}^2\gamma \operatorname{cn}\gamma \operatorname{dn}\gamma \,\mathrm{d}\gamma}{k^2 \operatorname{sn}^2\gamma^* \operatorname{sn}^2\gamma - 1}$$

since the integrand is an even function of γ. Introduce a new variable u, where $u = \gamma - K$; then using periodic properties we find

$$2\int_0^{2K} \frac{\operatorname{ns}^2\gamma \operatorname{cn}\gamma \operatorname{dn}\gamma \,\mathrm{d}\gamma}{k^2 \operatorname{sn}^2\gamma^* \operatorname{sn}^2\gamma - 1} = -2\int_{-K}^{K} \frac{k'^2 \operatorname{sn} u \operatorname{cd} u \,\mathrm{d}u}{k^2 \operatorname{sn}^2\gamma^* \operatorname{cd}^2 u - 1} = 0,$$

since the integrand is now an odd function of u. Further,

$$-k\,\mathrm{cn}\,\gamma^*\,\mathrm{dn}\,\gamma^*\int_{-2K}^{2K}\frac{\mathrm{d}\gamma}{1-k^2\,\mathrm{sn}^2\,\gamma^*\,\mathrm{sn}^2\,\gamma}$$

$$=-k\,\mathrm{cn}\,\gamma^*\,\mathrm{dn}\,\gamma^*\left\{\int_{-2K}^{2K}\mathrm{d}\gamma+k^2\,\mathrm{sn}^2\,\gamma^*\int_{-2K}^{2K}\frac{\mathrm{sn}^2\,\gamma\,\mathrm{d}\gamma}{1-k^2\,\mathrm{sn}^2\,\gamma^*\,\mathrm{sn}^2\,\gamma}\right\}$$

$$=-4Kk[\mathrm{cn}\,\gamma^*\,\mathrm{dn}\,\gamma^*+\mathrm{sn}\,\gamma^* Z(\gamma^*)],$$

from the definition of $Z(\gamma^*)$ (Byrd & Friedman 1954, p. 33). The result for I_2 may be obtained similarly.

The integrals of equation (37) for the lift are obtained as follows:

(i) $$\int_{-2K}^{2K}[1+\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma]\,\mathrm{d}\gamma=[\gamma+\mathrm{sn}\,\gamma]_{-2K}^{2K}=4K.$$

(ii) $$\int_{-2K}^{2K}\mathrm{sn}\,\gamma Z(\gamma)\,\mathrm{d}\gamma=2\int_{0}^{2K}\mathrm{sn}\,\gamma Z(\gamma)\,\mathrm{d}\gamma=2\int_{-K}^{K}\mathrm{cd}\,uZ(u)\,\mathrm{d}u-2k^2\int_{-K}^{K}\mathrm{sn}\,u\,\mathrm{cd}^2\,u\,\mathrm{d}u,$$

on putting $u=\gamma-K$, and using the periodic property for $Z(u+K)$. Thus

$$\int_{-2K}^{2K}\mathrm{sn}\,\gamma Z(\gamma)\,\mathrm{d}\gamma=0,$$

since the integrands are now all odd functions of u. The results (i) and (ii) are useful in § 14.

(iii) $$\int_{-2K}^{2K}\frac{\mathrm{sn}\,\gamma\,\mathrm{cn}\,\gamma^*\,\mathrm{dn}\,\gamma^*}{\mathrm{sn}\,\gamma^*-\mathrm{sn}\,\gamma}\,\mathrm{d}\gamma=\mathrm{cn}\,\gamma^*\,\mathrm{dn}\,\gamma^*\int_{-2K}^{2K}\frac{\mathrm{sn}^2\,\gamma\,\mathrm{d}\gamma}{\mathrm{sn}^2\,\gamma^*-\mathrm{sn}^2\,\gamma}$$

$$=-4K[\mathrm{cn}\,\gamma^*\,\mathrm{dn}\,\gamma^*+\mathrm{sn}\,\gamma^* Z(\gamma^*)],$$

on applying a formula given by Byrd & Friedman (1954, § 415·02).

(iv) $$\int_{-2K}^{2K}\frac{\mathrm{sn}\,\gamma\,\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma}{\mathrm{sn}\,\gamma^*-\mathrm{sn}\,\gamma}\,\mathrm{d}\gamma=-\int_{-2K}^{2K}\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma\,\mathrm{d}\gamma+\mathrm{sn}\,\gamma^*\int_{-2K}^{2K}\frac{\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma\,\mathrm{d}\gamma}{\mathrm{sn}\,\gamma^*-\mathrm{sn}\,\gamma}$$

$$=-[\mathrm{sn}\,\gamma]_{-2K}^{2K}-\mathrm{sn}\,\gamma^*[\ln(\mathrm{sn}\,\gamma^*-\mathrm{sn}\,\gamma)]_{-2K}^{2K}$$

$$=0.$$

The integrals in the moment equation (41) are:

(i) $\int_{-2K}^{2K}\ln\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)\mathrm{d}\gamma$ which vanishes after putting $u=\gamma-K$ as previously.

(ii) $$\int_{-2K}^{2K}\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma\ln\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)\mathrm{d}\gamma=\left[\mathrm{sn}\,\gamma\ln\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)\right]_{-2K}^{2K}-2k\int_{-2K}^{2K}\mathrm{sn}^2\,\gamma\,\mathrm{d}\gamma=\frac{8(E-K)}{k},$$

on integrating by parts and using the result (Byrd & Friedman 1954, p. 191)

$$\int_0^t\mathrm{sn}^2\,t\,\mathrm{d}t=\frac{1}{k^2}[t-E(t)].$$

(iii) $\int_{-2K}^{2K}\frac{\mathrm{sn}\,\gamma\,\mathrm{cn}\,\gamma^*\,\mathrm{dn}\,\gamma^*}{\mathrm{sn}\,\gamma^*-\mathrm{sn}\,\gamma}\ln\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)\mathrm{d}\gamma$ vanishes after putting $u=\gamma-K$.

(iv) If we write $$I'(\gamma^*)\equiv\int_{-2K}^{2K}\frac{\mathrm{sn}\,\gamma\,\mathrm{cn}\,\gamma\,\mathrm{dn}\,\gamma}{\mathrm{sn}\,\gamma^*-\mathrm{sn}\,\gamma}\ln\left(\frac{1-k\,\mathrm{cd}\,\gamma}{1+k\,\mathrm{cd}\,\gamma}\right)\mathrm{d}\gamma,$$

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then we find that

$$I'(\gamma^*) = \int_{-2K}^{2K} \frac{\operatorname{sn}^2\gamma \operatorname{cn}\gamma \operatorname{dn}\gamma}{\operatorname{sn}^2\gamma^* - \operatorname{sn}^2\gamma} \ln\left(\frac{1-k\operatorname{cd}\gamma}{1+k\operatorname{cd}\gamma}\right) \mathrm{d}\gamma$$

$$= -\int_{-2K}^{2K} \operatorname{cn}\gamma \operatorname{dn}\gamma \ln\left(\frac{1-k\operatorname{cd}\gamma}{1+k\operatorname{cd}\gamma}\right) \mathrm{d}\gamma + \operatorname{sn}^2\gamma^* \int_{-2K}^{2K} \frac{\operatorname{cn}\gamma \operatorname{dn}\gamma}{\operatorname{sn}^2\gamma^* - \operatorname{sn}^2\gamma} \ln\left(\frac{1-k\operatorname{cd}\gamma}{1+k\operatorname{cd}\gamma}\right) \mathrm{d}\gamma$$

$$= -\frac{8(E-K)}{k} + k\operatorname{sn}\gamma^* \int_{-2K}^{2K} \operatorname{sn}\gamma \ln\left(\frac{\operatorname{sn}\gamma^* - \operatorname{sn}\gamma}{\operatorname{sn}\gamma^* + \operatorname{sn}\gamma}\right) \mathrm{d}\gamma,$$

on integrating by parts. Consider now

$$J^* \equiv \int_{-2K}^{2K} \operatorname{sn}\gamma \ln\left(\frac{\operatorname{sn}\gamma^* - \operatorname{sn}\gamma}{\operatorname{sn}\gamma^* + \operatorname{sn}\gamma}\right) \mathrm{d}\gamma.$$

Then differentiating with respect to γ^*,

$$\frac{\partial J^*}{\partial \gamma^*} = 2\int_{-2K}^{2K} \frac{\operatorname{sn}^2\gamma \operatorname{cn}\gamma^* \operatorname{dn}\gamma^*}{\operatorname{sn}^2\gamma^* - \operatorname{sn}^2\gamma} \mathrm{d}\gamma = -8K[\operatorname{cn}\gamma^* \operatorname{dn}\gamma^* + \operatorname{sn}\gamma^* Z(\gamma^*)],$$

as previously. Thus $\quad J^* = -8K\left[\operatorname{sn}\gamma^* + \int_0^{\gamma^*} \operatorname{sn}\gamma^* Z(\gamma^*)\, \mathrm{d}\gamma^*\right],$

and so, after rearranging,

$$I'(\gamma^*) = \frac{8K}{k}\left[\operatorname{dn}^2\gamma^* - \frac{E}{K}\right] - 8Kk\operatorname{sn}\gamma^* \int_0^{\gamma^*} \operatorname{sn}\gamma^* Z(\gamma^*)\, \mathrm{d}\gamma^*.$$

This result is useful in the expansion of $I'(\gamma^*)$.

(v) If we write

$$I \equiv \int_{-2K}^{2K} \operatorname{sn}\gamma Z(\gamma) \ln\left(\frac{1-k\operatorname{cd}\gamma}{1+k\operatorname{cd}\gamma}\right) \mathrm{d}\gamma$$

then from the definition of $Z(\gamma)$ (Byrd & Friedman 1954, p. 230), we find

$$I = \frac{1}{4K}\int_{-2K}^{2K} \operatorname{sn}^2\gamma \operatorname{cn}\gamma \operatorname{dn}\gamma \ln\left(\frac{1-k\operatorname{cd}\gamma}{1+k\operatorname{cd}\gamma}\right) \int_{-2K}^{2K} \frac{\mathrm{d}u}{\operatorname{sn}^2 u - \operatorname{sn}^2\gamma} \mathrm{d}\gamma$$

$$= \frac{1}{4K}\int_{-2K}^{2K} I'(u)\, \mathrm{d}u. \tag{44}$$

Appendix 3. An expression for the pressure distribution

On rearranging terms we can write equation (30) for the velocity distribution on the aerofoil surface

$$\Omega(\gamma) = \frac{1}{2\pi}\int_{-2K}^{2K} \theta^* \frac{\operatorname{cn}\gamma^* \operatorname{dn}\gamma^* + \operatorname{cn}\gamma \operatorname{dn}\gamma}{\operatorname{sn}\gamma^* - \operatorname{sn}\gamma} \mathrm{d}\gamma^* + \frac{1}{2\pi}\int_0^{K'} \mathrm{X} \frac{\operatorname{sn}^2 \mathrm{i}\eta^* (\operatorname{cn}\gamma \operatorname{dn}\gamma - \operatorname{cn}\mathrm{i}\eta^* \operatorname{dn}\mathrm{i}\eta^*)}{\operatorname{sn}\gamma(\operatorname{sn}^2\mathrm{i}\eta^* - \operatorname{sn}^2\gamma)} \mathrm{d}\eta^*$$

$$- \frac{\operatorname{cn}\gamma \operatorname{dn}\gamma}{2\pi \operatorname{sn}\gamma}\int_0^{K'} \mathrm{X}\, \mathrm{d}\eta^* + \frac{1}{2\pi \operatorname{sn}\gamma}\int_0^{K'} \mathrm{X} \operatorname{cn}\mathrm{i}\eta^* \operatorname{dn}\mathrm{i}\eta^*\, \mathrm{d}\eta^*. \tag{136}$$

Replacing the last two integrals of this from equations (9) and (17) respectively, and simplifying, we find for (136)

$$\Omega(\gamma) = \frac{1}{2\pi}\int_{-2K}^{2K} \theta^* \left\{\frac{\operatorname{sn}\gamma^*(\operatorname{cn}\gamma^* \operatorname{dn}\gamma^* + \operatorname{cn}\gamma \operatorname{dn}\gamma)}{\operatorname{sn}\gamma(\operatorname{sn}\gamma^* - \operatorname{sn}\gamma)} + \frac{\operatorname{sn}\gamma^* Z(\gamma^*)}{\operatorname{sn}\gamma}\right\} \mathrm{d}\gamma^*$$

$$+ \frac{1}{2\pi}\int_0^{K'} \mathrm{X} \left\{\frac{\operatorname{sn}^2\mathrm{i}\eta^*(\operatorname{cn}\gamma \operatorname{dn}\gamma - \operatorname{cn}\mathrm{i}\eta^* \operatorname{dn}\mathrm{i}\eta^*)}{\operatorname{sn}\gamma(\operatorname{sn}^2\mathrm{i}\eta^* - \operatorname{sn}^2\gamma)} - \frac{\operatorname{sn}\mathrm{i}\eta^* Z(\mathrm{i}\eta^*)}{\operatorname{sn}\gamma}\right\} \mathrm{d}\eta^*.$$

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From this we have

$$\operatorname{sn}\gamma\frac{\partial\Omega}{\partial t}=\frac{1}{2\pi}\int_{-2K}^{2K}\dot{\theta}^*\left\{\frac{\operatorname{sn}\gamma^*(\operatorname{cn}\gamma^*\operatorname{dn}\gamma^*+\operatorname{cn}\gamma\operatorname{dn}\gamma)}{\operatorname{sn}\gamma^*-\operatorname{sn}\gamma}+\operatorname{sn}\gamma^*Z(\gamma^*)\right\}\mathrm{d}\gamma^*$$
$$+\frac{1}{2\pi}\int_0^{K'}\frac{\partial X}{\partial t}\left\{\frac{\operatorname{sn}^2\mathrm{i}\eta^*(\operatorname{cn}\gamma\operatorname{dn}\gamma-\operatorname{cn}\mathrm{i}\eta^*\operatorname{dn}\mathrm{i}\eta^*)}{\operatorname{sn}^2\mathrm{i}\eta^*-\operatorname{sn}^2\gamma}-\operatorname{sn}\mathrm{i}\eta^*Z(\mathrm{i}\eta^*)\right\}\mathrm{d}\eta^*. \quad (137)$$

The differential equation (23) enables the latter integral of (137) to be expressed as

$$I\equiv-\frac{U}{4hk}\int_0^{K'}\mathrm{i}\frac{\partial X}{\partial\eta^*}\left\{\frac{\operatorname{sn}\mathrm{i}\eta^*(\operatorname{cn}\gamma\operatorname{dn}\gamma-\operatorname{cn}\mathrm{i}\eta^*\operatorname{dn}\mathrm{i}\eta^*)}{\operatorname{sn}^2\mathrm{i}\eta^*-\operatorname{sn}^2\gamma}-Z(\mathrm{i}\eta^*)\right\}\mathrm{d}\eta^*$$
$$=\frac{U}{4hk}\int_0^{K'}\mathrm{i}X\frac{\partial}{\partial\eta^*}\left\{\frac{\operatorname{sn}\mathrm{i}\eta^*(\operatorname{cn}\gamma\operatorname{dn}\gamma-\operatorname{cn}\mathrm{i}\eta^*\operatorname{dn}\mathrm{i}\eta^*)}{\operatorname{sn}^2\mathrm{i}\eta^*-\operatorname{sn}^2\gamma}-Z(\mathrm{i}\eta^*)\right\}\mathrm{d}\eta^*,$$

after integration by parts—the integrated term vanishing. Straightforward differentiation now shows that

$$\mathrm{i}\frac{\partial}{\partial\eta}\left\{\frac{\operatorname{sn}\mathrm{i}\eta(\operatorname{cn}\gamma\operatorname{dn}\gamma-\operatorname{cn}\mathrm{i}\eta\operatorname{dn}\mathrm{i}\eta)}{\operatorname{sn}^2\mathrm{i}\eta-\operatorname{sn}^2\gamma}-Z(\mathrm{i}\eta)\right\}=-\frac{\partial}{\partial\gamma}\left\{\frac{\operatorname{sn}\gamma(\operatorname{cn}\gamma\operatorname{dn}\gamma-\operatorname{cn}\mathrm{i}\eta\operatorname{dn}\mathrm{i}\eta)}{\operatorname{sn}^2\mathrm{i}\eta-\operatorname{sn}^2\gamma}-Z(\gamma)\right\}.$$

Therefore (137) may be written

$$\operatorname{sn}\gamma\frac{\partial\Omega}{\partial t}=\frac{1}{2\pi}\int_{-2K}^{2K}\dot{\theta}^*\left\{\frac{\operatorname{sn}\gamma^*(\operatorname{cn}\gamma^*\operatorname{dn}\gamma^*+\operatorname{cn}\gamma\operatorname{dn}\gamma)}{\operatorname{sn}\gamma^*-\operatorname{sn}\gamma}+\operatorname{sn}\gamma^*Z(\gamma^*)\right\}\mathrm{d}\gamma^*$$
$$-\frac{U}{4hk}\int_0^{K'}X\frac{\partial}{\partial\gamma}\left\{\frac{\operatorname{sn}\gamma(\operatorname{cn}\gamma\operatorname{dn}\gamma-\operatorname{cn}\mathrm{i}\eta^*\operatorname{dn}\mathrm{i}\eta^*)}{\operatorname{sn}^2\mathrm{i}\eta^*-\operatorname{sn}^2\gamma}-Z(\gamma)\right\}\mathrm{d}\eta^*;$$

and so

$$\frac{2hk\rho U}{\pi}\int_0^{\gamma}\operatorname{sn}\gamma\frac{\partial\Omega}{\partial t}\mathrm{d}\gamma=\frac{hk\rho U}{\pi^2}\int_{-2K}^{2K}\dot{\theta}^*\int_0^{\gamma}\left\{\frac{\operatorname{sn}\gamma^*(\operatorname{cn}\gamma^*\operatorname{dn}\gamma^*+\operatorname{cn}\gamma\operatorname{dn}\gamma)}{\operatorname{sn}\gamma^*-\operatorname{sn}\gamma}+\operatorname{sn}\gamma^*Z(\gamma^*)\right\}\mathrm{d}\gamma\,\mathrm{d}\gamma^*$$
$$-\frac{\rho U^2}{2\pi}\int_0^{K'}X\left\{\frac{\operatorname{sn}\gamma(\operatorname{cn}\gamma\operatorname{dn}\gamma-\operatorname{cn}\mathrm{i}\eta^*\operatorname{dn}\mathrm{i}\eta^*)}{\operatorname{sn}^2\mathrm{i}\eta^*-\operatorname{sn}^2\gamma}-Z(\gamma)\right\}\mathrm{d}\eta^*. \quad (138)$$

By combining this equation with the term $\rho U^2\Omega$ from (30), the pressure equation (31) can now be written down.

Introduction of the function $G(\gamma^*)$, defined in equation (32), allows the second integral of (31) to be expressed as

$$I\equiv\frac{\rho U}{2\pi}\int_0^{\gamma}\int_{-2K}^{2K}\frac{\partial}{\partial\gamma^*}\dot{G}(\gamma^*)\left\{\frac{\operatorname{cn}\gamma^*\operatorname{dn}\gamma^*+\operatorname{cn}\gamma\operatorname{dn}\gamma}{\operatorname{sn}\gamma^*-\operatorname{sn}\gamma}+Z(\gamma^*)\right\}\mathrm{d}\gamma^*\,\mathrm{d}\gamma,$$

which becomes, on integrating by parts,

$$I=\frac{\rho U}{2\pi}\int_0^{\gamma}\left[\dot{G}(\gamma^*)\left\{\frac{\operatorname{cn}\gamma^*\operatorname{dn}\gamma^*+\operatorname{cn}\gamma\operatorname{dn}\gamma}{\operatorname{sn}\gamma^*-\operatorname{sn}\gamma}+Z(\gamma^*)\right\}\right]_{-2K}^{2K}\mathrm{d}\gamma$$
$$-\frac{\rho U}{2\pi}\int_0^{\gamma}\int_{-2K}^{2K}\dot{G}(\gamma^*)\frac{\partial}{\partial\gamma^*}\left\{\frac{\operatorname{cn}\gamma^*\operatorname{dn}\gamma^*+\operatorname{cn}\gamma\operatorname{dn}\gamma}{\operatorname{sn}\gamma^*-\operatorname{sn}\gamma}+Z(\gamma^*)\right\}\mathrm{d}\gamma^*\,\mathrm{d}\gamma.$$

From equations (10) and (32) it follows that

$$G(2K)-G(-2K)=0,$$

and so the integrated term of I vanishes. Further it can be shown that

$$\frac{\partial}{\partial\gamma^*}\left\{\frac{\operatorname{cn}\gamma^*\operatorname{dn}\gamma^*+\operatorname{cn}\gamma\operatorname{dn}\gamma}{\operatorname{sn}\gamma^*-\operatorname{sn}\gamma}+Z(\gamma^*)\right\}=-\frac{\partial}{\partial\gamma}\left\{\frac{\operatorname{cn}\gamma^*\operatorname{dn}\gamma^*+\operatorname{cn}\gamma\operatorname{dn}\gamma}{\operatorname{sn}\gamma^*-\operatorname{sn}\gamma}-Z(\gamma)\right\}.$$

Thus we may write

$$I=\frac{\rho U}{2\pi}\int_{-2K}^{2K}\dot{G}(\gamma^*)\left[\frac{\operatorname{cn}\gamma^*\operatorname{dn}\gamma^*+\operatorname{cn}\gamma\operatorname{dn}\gamma}{\operatorname{sn}\gamma^*-\operatorname{sn}\gamma}-Z(\gamma)\right]\mathrm{d}\gamma^*.$$

Substituting this back into (31), we finally obtain the equation (33).

Appendix 4. Transformation of the function $T(\lambda, r)$

From equation (83) and the definition of $\mathcal{Q}(\cosh r)$, it is found that $T(\lambda, r)$ may be written

$$T(\lambda,r)=\frac{NQ_{N-1}-(N+1)\,Q_{N+1}+\left(\dfrac{2E}{Kk'^2}-1-\dfrac{4\mathrm{i}\lambda k}{rk'^2}\right)Q_N}{NQ_{N-1}-(N+1)\,Q_{N+1}+\left(\dfrac{2E}{Kk'^2}-1+\dfrac{4\mathrm{i}\lambda k}{rk'^2}\right)Q_N}.$$

Whittaker & Watson (1946, p. 317) has the formula

$$Q_N(z)=\frac{\pi^{\frac{1}{2}}}{2^{N+1}}\frac{\Gamma(N+1)}{\Gamma(N+\frac{3}{2})}\frac{1}{z^{N+1}}F\left[\tfrac{1}{2}N+\tfrac{1}{2},\tfrac{1}{2}N+1;\,N+\tfrac{3}{2};\,\frac{1}{z^2}\right],$$

where $F[a,b;c,u]$ is the hypergeometric function, and $\Gamma(u)$ the gamma function. Using the transformation formula (Erdelyi 1953, p. 112),

$$F[a,a+\tfrac{1}{2};c;u]=(1+u^{\frac{1}{2}})^{-2a}F[2a,c-\tfrac{1}{2};2c-1;2u^{\frac{1}{2}}(1+u^{\frac{1}{2}})^{-1}],$$

we obtain $$Q_N(z)=\frac{\pi^{\frac{1}{2}}}{2^{N+1}}\frac{\Gamma(N+1)}{\Gamma(N+\frac{3}{2})}(1+z)^{-N-1}F\left[N+1,N+1;\,2N+2;\,\frac{2}{1+z}\right],$$

where we have put $$a=\tfrac{1}{2}N+\tfrac{1}{2},\quad c=N+\tfrac{3}{2},\quad u=\frac{1}{z^2}.$$

Further, Erdelyi has the result (p. 113)

$$F[a,a;2a;u]=[\tfrac{1}{2}+\tfrac{1}{2}(1-u)^{\frac{1}{2}}]^{-2a}F\left[a,\tfrac{1}{2};\,a+\tfrac{1}{2};\left\{\frac{1-(1-u)^{\frac{1}{2}}}{1+(1-u)^{\frac{1}{2}}}\right\}^2\right].$$

Thus on putting $u=\dfrac{2}{1+z}$, $a=N+1$, we obtain

$$Q_N(z)=\frac{\pi^{\frac{1}{2}}}{2^{N+1}}\frac{\Gamma(N+1)}{\Gamma(N+\frac{3}{2})}(1+z)^{-N-1}$$
$$\times\left[\frac{1}{2}\left(1+\frac{\surd(z-1)}{\surd(z+1)}\right)\right]^{-2N-2}F\left[N+1,\tfrac{1}{2};\,N+\tfrac{3}{2};\left\{\frac{\surd(z+1)-\surd(z-1)}{\surd(z+1)+\surd(z-1)}\right\}^2\right].$$

Writing $z=\cosh r$, we find this to be

$$Q_N(\cosh r)=\pi^{\frac{1}{2}}\frac{\Gamma(N+1)}{\Gamma(N+\frac{3}{2})}\mathrm{e}^{-(N+1)r}F[N+1,\tfrac{1}{2};\,N+\tfrac{3}{2};\,\mathrm{e}^{-2r}].$$

Application of this formula, with N taking consecutive values $N=\mathrm{i}\lambda/r-\frac{1}{2}$, $N=\mathrm{i}\lambda/r-\frac{3}{2}$, $N=\mathrm{i}\lambda/r+\frac{1}{2}$, immediately yields the form (128).